Apollo Launch Vehicles

Saturn IB  
Saturn V
Evolution of Saturn launch vehicles
Development of rocket motors with high thrust
F-1 and J-2
Von Braun group used to developing integrated vehicle and payload; now different design teams would be involved
Contending styles and approaches
Grumman: prime contractor for LEM
Expansion of MSFC facilities
• MSFC’s “factory look” inherited from Army arsenal
• Chrysler, Boeing, North American: Saturn contractors
• Description of extensive facilities for production
• Flight worthiness certificates
• GE: facilities management
• Rocketdyne: F-1 engine contractor; thrust vectoring
• Separate tanks, fuel slosh baffles
• Study of 1st stage recovery

... and Saturn V 3rd stage
• Improbable estimate of Saturn launch rates (100/year)
• Block I and Block II concept
• Loads and stiffness of structure
• Spider beams
• Structure, stabilizing fins
• Flight stability, low natural frequency, advanced control
• Aerodynamic issues
• Acoustic vibration and impact
• Douglas: S-IV engine contractor

Conventional Cryogenics: H-1 and F-1

• Cryogenic technology
  • Apparently overlooked Goddard’s work, referred to German experience
• Saturn engine antecedents
  • Development relied on IRBM/ICBM experience
• LOX/RP-1 (Kerosene) propellants
• H-1 powered the Saturn I and IB
  • Upgrade of existing engine for Thor-Jupiter
• F-1 powered the Saturn V
• H-1 milestones and facilities
• General description
• Development problems
• Testing and controlling combustion instability
• *S-IC and the Huntsville connection*
• *Tools and tankage*
• *Few but cumbersome components*
• *Fabrication and manufacture*
• **Origins of the F-1**
  - Air Force legacy (1955)
  - Design undertaken before vehicle or mission were identified
• **Big engine, big problems**
  - 16:1 nozzle expansion
  - 6.67 MN thrust
• **F-1 injector**
• **Combustion instability**
  - Theoretical work by Luigi Crocco and David Harrje, Princeton

• **F-1 turbopumps**
  - Oxygen: 24,811 gal/min
  - RP-1: 15,741 gal/min
• **Thrust chamber and furnace brazing**
• **Other components and subsystems**
• **Static test to flight test**
F-1 Engine
Start from Command Module

- Thrust chambers
  - Double-wall construction
- Exhaust nozzles
  - Bell shape improved flow
- Turbopumps
  - Problems with turbine blades, bearings, cavitation
- Packaging and system design
  - Gimbaling for thrust vector control
- Engine problem phases
Unconventional Cryogenics: RL-10 and J-2

- Tsiolkovsky proposed LOX/LH2 propellants
- Early testing
- Centaur upper stage
- 6-RL-10 engines used on Saturn 1 S-IV stage
- Succeeded by one J-2 engine on SIVB
S-IVB with single J-2 engine
From the S-IV to the S-IVB

- 1st stage of the Saturn V to be designed
- Started before von Braun group transferred
- Rationale for choosing Douglas to build S-IV and S-IVB
• Coasting orbit, restart capability
• Launch window expansion
• Volumetric considerations for hydrogen and oxygen tank positions
• Fabrication and manufacturing
• “Waffle-shaped” integral stiffening of the skin
• Reliance on aviation production technology
• Ullage rockets? Retro-rockets?
• Domes and hemispheric common bulkhead
• Thin aluminum shells
• Self-supporting structures

• Proportions of the stages
• Both used brand new technology extensively
  • Welding, forging, materials
• S-IC off to a better start
• S-II left to make up for end-game problems, as S-IC and S-IVB were further along in development
• *S-II Concepts*
• *Configuration*
• *Systems*
• *Trial and error: the welding problem*
• *Management difficulties at NAA*
• *Importance of NASA HQ intervention*

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**Saturn Checkout – Control Center**

[Diagram of Saturn Checkout – Control Center]
STAGES TO SATURN

The Quintessential Computer

- Automatic checkout
- Launch sequencing and control
- Launch vehicle guidance and control

Saturn ST-124 Inertial Measurement Unit
Launch Vehicle Design

Launch Vehicle Configuration Design Goals

- Minimum weight -> sphere
- Minimum drag -> slender body
- Minimum axial load -> low thrust
- Minimum gravity loss -> high thrust
- Maximum payload -> lightweight structure, high mass ratio, multiple stages, high specific impulse
- Perceived simplicity, improved range safety -> single stage
- Minimum cost -> low-cost materials, economies of scale
- Minimum environmental impact -> non-toxic propellant
**Typical Payload Ratios, $\lambda_{overall}$ for Launch to Low Earth Orbit (LEO) without SRBs**

- Ariane 5: 0.021  
- Atlas V: 0.027  
- Delta IV: 0.034  
- Epsilon: 0.014  
- Falcon 9: 0.026  
- GSLV: 0.012  
- H-IIA: 0.035  
- Long March 3: 0.025  
- Minotaur IV: 0.02  
- Proton: 0.029  
- PSLV: 0.011  
- Saturn IB: 0.035  
- Scout: 0.008  
- Shahab: 0.002  
- Shavit: 0.011  
- Sputnik R-7: 0.002  
- Taurus: 0.018  
- Vega: 0.011  
- Voshkod: 0.020  
- Zenit: 0.031

---

**Specific Impulse of Launch Vehicle Powerplants**
Vertical Takeoff, Horizontal Landing Vehicles

- Martin Astro-Rocket
- General Dynamics Triamese

- Heat shield-to-heat shield
- Three “identical” parallel stages

Horizontal vs. Vertical Launch

- Feasibility of “airline-like” operations?
- Use of high $I_{sp}$ air-breathing engines
- Rocket stages lifted above the sensible atmosphere
- Flexible launch location, direction, and time
Specific Energy Contributed in Boost Phase

- **Total Energy** = Kinetic plus Potential Energy (relative to flat earth)
  \[ E = \frac{mV^2}{2} + mgh \]

- **Specific Total Energy** = Energy per unit weight = Energy Height (km)
  \[ E' = \frac{V^2}{2g} + h \]

### Specific Energy Contributed in Boost Phase

- **Specific Energy contributed by 1st stage of launch vehicle**
  - Less remaining drag loss (typical)
  - Plus Earth’s rotation speed (typical)

<table>
<thead>
<tr>
<th></th>
<th>Altitude, km</th>
<th>Mach Number</th>
<th>Earth RelativeVelocity, km/s</th>
<th>Remaining Drag Loss, km/s</th>
<th>Earth Rotation Speed, km/s</th>
<th>Specific Kinetic Energy, km</th>
<th>Total Specific Energy, km</th>
<th>Percent of Goal</th>
</tr>
</thead>
<tbody>
<tr>
<td>Scout 1st-Stage Burnout</td>
<td>22</td>
<td>4</td>
<td>1.2</td>
<td>0.05</td>
<td>0.4</td>
<td>123.42</td>
<td>145.42</td>
<td>3.93%</td>
</tr>
<tr>
<td>Subsonic Horizontal Launch</td>
<td>12</td>
<td>0.8</td>
<td>0.235</td>
<td>0.15</td>
<td>0.4</td>
<td>12.05</td>
<td>24.05</td>
<td>0.65%</td>
</tr>
<tr>
<td>Supersonic Horizontal Launch</td>
<td>25</td>
<td>3</td>
<td>0.93</td>
<td>0.04</td>
<td>0.4</td>
<td>85.57</td>
<td>110.57</td>
<td>2.99%</td>
</tr>
<tr>
<td>Scramjet Horizontal Launch</td>
<td>50</td>
<td>12</td>
<td>3.6</td>
<td>0</td>
<td>0.4</td>
<td>829.19</td>
<td>879.19</td>
<td>23.74%</td>
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<tr>
<td>Target Orbit</td>
<td>300</td>
<td>25</td>
<td>7.4</td>
<td>0</td>
<td>0.4</td>
<td>3403.34</td>
<td>3703.34</td>
<td></td>
</tr>
</tbody>
</table>
Aerospace Planes (Trans-Atmospheric Vehicles)

- **Power for takeoff**
  - Turbojet/fans
  - Multi-cycle air-breathing engines
  - Rockets
- **Single-stage-to-orbit**
  - Carrying dead weight into orbit
  - High structural ratio for wings, powerplants, and reusability
  - SSTO has very low payload ratio

Venture Star/X-33

- Venture Star: Reusable, single-stage-to-orbit, proposed Space Shuttle replacement
- X-33: Sub-orbital test vehicle
- Improved thermal protection system (compared to SSV)
- Linear spike nozzle rocket (altitude compensation)
- Cancelled following tank failure in testing
Trans-Atmospheric Vehicle Concepts

- Various approaches to staging
  - Boeing TAV
  - Rockwell TAV
  - Rockwell StarRaker
  - Lockheed Clipper
  - Lockheed TAV

Pegasus Air-Launched Rocket

- Initial mass: 23,000 kg
- Payload mass to LEO: 440 kg
- Payload ratio = 0.019 (rocket)
  = 0.001 (airplane)

- Three solid-rocket stages launched from an aircraft
- Aerodynamic lift used to rotate vehicle for climb
Stratolaunch

- Based on two Boeing 747-400 airplanes
  - Six turbofan engines
  - Wingspan: 117 m
  - Takeoff mass: 540,000 kg
- Pegasus II rocket: 210,000 kg
- Payload to LEO: 6,100 kg
- Payload ratio = 0.029 (rocket) = 0.011 (airplane)
- First flight: April 14, 2019

Launch Vehicle Structural Loads

- Static/quasi-static loads
  - Gravity and thrust
  - Propellant tank internal pressure
  - Thermal effects
    - Rocket
    - Cryogenic propellant
    - Aerodynamic heating
- Dynamic loads
  - Bending and torsion
  - “Pogo” oscillations
  - Fuel sloshing
  - Aerodynamics and thrust vectoring
- Acoustic and mechanical vibration loads
  - Rocket engine
  - Aerodynamic noise
Structural Material Properties

- **Stress**, $\sigma$: Force per unit area
- **Strain**, $\varepsilon$: Elongation per unit length
- **Proportionality factor**, $E$: Modulus of elasticity, or Young’s modulus
- **Strain deformation is reversible below the elastic limit**
- **Elastic limit = yield strength**
- **Proportional limit ill-defined for many materials**
- **Ultimate stress**: Material breaks

\[ \sigma = E \varepsilon \]

![Graph showing stress-strain relationship](image)

Poisson’s ratio, $\nu$:

\[ \nu = \frac{\varepsilon_{\text{axial}}}{\varepsilon_{\text{axial}}} \]

Typically $0.1$ to $0.35$

Thickening under compression

Thinning under tension

Uniform Stress Conditions

- **Average axial stress**, $\sigma$

\[ \sigma = \frac{P}{A} = \text{Load}/\text{Cross Sectional Area} \]

- **Average axial strain**, $\varepsilon$

\[ \varepsilon = \frac{\Delta L}{L} \]

- **Effective spring constant**, $k_s$

\[ \sigma = \frac{P}{A} = E\varepsilon = E \frac{\Delta L}{L} \]

\[ P = \frac{AE}{L} \Delta L = k_s \Delta L \]
## Structural Material Properties

<table>
<thead>
<tr>
<th>Material Properties (Wikipedia)</th>
<th>Young's Modulus, GPa</th>
<th>Elastic Limit, MPa</th>
<th>Density, g/cm³</th>
</tr>
</thead>
<tbody>
<tr>
<td>Aluminum Alloy</td>
<td>69</td>
<td>400</td>
<td>2.7</td>
</tr>
<tr>
<td>Carbon-Fiber Composite</td>
<td>530</td>
<td>-</td>
<td>1.8</td>
</tr>
<tr>
<td>Fiber-Glass Composite</td>
<td>125-150</td>
<td>-</td>
<td>2.5</td>
</tr>
<tr>
<td>Magnesium</td>
<td>45</td>
<td>100</td>
<td>1.7</td>
</tr>
<tr>
<td>Steel</td>
<td>200</td>
<td>250-700</td>
<td>7.8</td>
</tr>
<tr>
<td>Titanium</td>
<td>105-120</td>
<td>830</td>
<td>4.5</td>
</tr>
</tbody>
</table>

### Structural Stiffness

- **Geometric stiffness of structure that bends about its x axis portrayed by its area moment of inertia**

  \[ I_x = \int_{z_{\text{min}}}^{z_{\text{max}}} x(z) z^2 \, dz \]

- **Area moment of inertia for simple cross-sections**
  - **Solid rectangle of height, h, and width, w:**
    \[ I_x = \frac{wh^3}{12} \]
  - **Solid circle of radius, r:**
    \[ I_x = \frac{\pi r^4}{4} \]
  - **Circular cylindrical tube with inner radius, r_i, and outer radius, r_o:**
    \[ I_x = \frac{\pi (r_o^4 - r_i^4)}{4} \]
**Structural Stiffeners**

- Axial stiffeners: high $I_x$ per cross-sectional area
- Circular stiffeners increase resistance to buckling
- Honeycomb and waffled surfaces remove weight while retaining $I_x$

**Propellant Tank Configurations for Launch Vehicles**

- Serial tanks with common bulkhead
- Separate serial tanks
- Parallel tanks
Mercury-Redstone Structure

Semi-monocoque structure (load-bearing skin stiffened by internal components)
External skin, internal tanks separated by longerons and circular stiffeners
Aerodynamic and exhaust vanes for thrust vectoring

Hoop, Axial, and Radial Stresses in Pressurized, Thin-Walled Cylindrical Tanks

For the cylinder
\[
\sigma_{\text{hoop}} = \frac{pR}{T} \\
\sigma_{\text{axial}} = \frac{pR}{2T} \\
\sigma_{\text{radial}} = \text{negligible}
\]

For the spherical end cap
\[
\sigma_{\text{hoop}} = \sigma_{\text{axial}} = \frac{pR}{2T} \\
\sigma_{\text{radial}} = \text{negligible}
\]

Cylinder hoop stress is limiting factor, \( \sigma_{\text{hoop}} > \sigma_{\text{axial}} \)
Weight Comparison of Thin-Walled Spherical and Cylindrical Tanks

(Sechler, in Space Technology, 1959)

- Pressure vessels have same volume and same maximum shell stresses due to internal pressure; hydraulic head* neglected
  - $R_c$ = cylindrical radius
  - $R_s$ = spherical radius
- Weight increases as L/D increases

* Hydraulic head = Liquid pressure per unit of weight x load factor

Staged Spherical vs. Cylindrical Tanks

(Sechler, in Space Technology, 1959)

Comparison: pressure vessels have same volume and same maximum shell stresses due to internal pressure with and without hydraulic head (with full tanks)

Numerical example for load factor of 2.5
Cylindrical tanks lighter than comparable spherical tanks
Common bulkead even lighter
Critical Axial Stress in Thin-Walled Cylinders
(Sechler, in Space Technology, 1959)

- Compressive axial stress can lead to buckling failure
- Critical stress, $\sigma_c$, can be increased by
  - Increasing $E$
  - Increasing wall thickness, $t$
     - solid material
     - honeycomb
  - Adding rings to decrease effective length
  - Adding longitudinal stringers
  - Fixing axial boundary conditions
  - Pressurizing the cylinder

$$\frac{\sigma_c}{E} = (K_o + K_p) \left( \frac{t}{R} \right)^{1.3} + 0.16 \left( \frac{t}{L} \right)^{1.6}$$

SM-65/Mercury Atlas

- Launch vehicle designed with “balloon” propellant tanks to save weight
  - Monocoque design (no internal bracing or stiffening)
  - Stainless steel skin 0.1- to 0.4-in thick
  - Vehicle would collapse without internal pressurization
  - Filled with nitrogen at 5 psi when not fuelled

- With internal pressure

$$\frac{\sigma_c}{E} = (K_o + K_p) \left( \frac{t}{R} \right)$$

where

$$K_o = 9 \left( \frac{t}{R} \right)^{0.6} + 0.16 \left( \frac{R}{L} \right)^{1.3} \left( \frac{t}{R} \right)^{0.3}$$

$$K_p = 0.191 \left( \frac{P}{E} \right) \left( \frac{R}{t} \right)^2$$
Force and Moments on a Slender Cantilever (Fixed-Free) Beam

- Idealization of
  - Launch vehicle tied-down to a launch pad
  - Structural member of a payload

- For a point force
  - Force and moment must be opposed at the base
  - Shear distribution is constant
  - Bending moment increases as moment arm increases
  - Torsional moment and moment arm are fixed

Bending Stiffness

- Neutral axis neither shrinks nor stretches in bending
- For small deflections, the bending radius of curvature of the neutral axis is
  \[ r = \frac{EI}{M} \]

Deflection at a point characterized by displacement and angle:
**Bending Deflection**

Second derivative of $z$ and first derivative of $\phi$ are inversely proportional to the bending radius:

$$\frac{d^2 z}{dx^2} = \frac{d\phi}{dx} = \frac{M_y}{EI_y}$$

**Buckling**

- Predominant steady stress during launch is compression
- Thin columns, plates, and shells are subject to elastic instability in compression
- Buckling can occur below the material’s elastic limit

Critical buckling stress of a column (Euler equation)

$$\sigma_{cr} = \frac{C\pi^2 E}{(L / \rho)^2} = \frac{P}{A}$$

* $C$ = function of end “fixity”
* $E$ = modulus of elasticity
* $L$ = column length
* $\rho = \sqrt{I/A}$ = radius of gyration
* $P_{cr} = $ critical buckling load
* $A = $ cross sectional area
Effect of “Fixity” on Critical Loads for Beam Buckling

- Euler equation
  - Slender columns
  - Critical stress below the elastic limit
  - Relatively thick column walls
- Local collapse due to thin walls is called crippling

\[ P_{cr} = \frac{\pi^2 EI}{4L^2} \]
\[ P_{cr} = \frac{\pi^2 EI}{L^2} \]
\[ P_{cr} = \frac{4\pi^2 EI}{(0.71L)^2} \]

Quasi-Static Loads
(Spacecraft Systems Engineering, 2003)

<table>
<thead>
<tr>
<th>Flight event</th>
<th>Acceleration (g) Q.S.L.</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Longitudinal</td>
</tr>
<tr>
<td>Maximum dynamic pressure</td>
<td>-3.0</td>
</tr>
<tr>
<td>Before thrust termination</td>
<td>-5.5</td>
</tr>
<tr>
<td>During thrust tail-off</td>
<td>+2.5</td>
</tr>
</tbody>
</table>

Note: The minus sign with longitude axis values indicates compression.
**Bending Moment due to a Distributed Normal Force**

Flight through varying winds produces vibratory bending input

\[ N'_y(x_a) \]

---

**Pogo Oscillation**

- Longitudinal resonance of the launch vehicle structure
  - Flexing of the propellant-feed pipes induces thrust variation
- Gas-filled cavities added to pipes, damping oscillation
- Problem experienced on Saturn 5, Titan 2, (±2.5 g), other vehicles
**Fuel Slosh**

- Lateral motion of liquid propellant in partially empty tank induces inertial forces
- Resonance with flight motions can occur
- Problem reduced by baffles

**Springs and Dampers**

Force due to linear spring

\[ f_s = -k_s \Delta x = -k_s (x - x_0) \quad ; \quad k_s = \text{spring constant} \]

Force due to linear damper

\[ f_d = -k_d \Delta \dot{x} = -k_d \Delta v = -k_d (\dot{v} - \dot{v}_0) \quad ; \quad k_d = \text{damping constant} \]
Mass, Spring, and Damper

Newton’s second law leads to a second-order dynamic system

\[ \Delta \ddot{x} = f_x / m = (-k_d \Delta \dot{x} - k_s \Delta x + \text{forcing function}) / m \]

\[ \Delta \ddot{x} + \frac{k_d}{m} \Delta \dot{x} + \frac{k_s}{m} \Delta x = \frac{\text{forcing function}}{m} \]

\[ \Delta \ddot{x} + 2\zeta \omega_n \Delta \dot{x} + \omega_n^2 \Delta x = \omega_n^2 \Delta u \]

\( \omega_n = \text{natural frequency, rad/s} \)
\( \zeta = \text{damping ratio} \)
\( \Delta x = \text{displacement} \)
\( \Delta u = \text{disturbance or control} \)

Response to Initial Condition

- **Lightly damped system** has a decaying, oscillatory transient response
- **Forcing by step or impulse** produces a similar transient response

\( \omega_n = 6.28 \text{ rad/sec} \)
\( \zeta = 0.05 \)
Oscillations

\[ \Delta x = A \sin(\omega t) \]

\[ \Delta \dot{x} = A \omega \cos(\omega t) = A \omega \sin(\omega t + \pi/2) \]

\[ \Delta \ddot{x} = -A \omega^2 \sin(\omega t) = A \omega^2 \sin(\omega t + \pi) \]

- Phase angle of velocity (wrt displacement) is \( \pi/2 \) rad (or 90°)
- Phase angle of acceleration is \( \pi \) rad (or 180°)
- As oscillation frequency, \( \omega \) varies
  - Velocity amplitude is proportional to \( \omega \)
  - Acceleration amplitude is proportional to \( \omega^2 \)

Frequency Response of the 2\textsuperscript{nd}-Order System

- Convenient to plot response on logarithmic scale
  \[ \ln[A(\omega)e^{j\phi(\omega)}] = \ln A(\omega) + j\phi(\omega) \]

- Bode plot
  - 20 log(Amplitude Ratio) [dB] vs. log \( \omega \)
  - Phase angle (deg) vs. log \( \omega \)

- Natural frequency characterized by
  - Peak (resonance) in amplitude response
  - Sharp drop in phase angle

- Acceleration frequency response peak occurs at natural frequency
Acceleration Response of the 2nd-Order System

- Low-frequency acceleration response is attenuated
- Sinusoidal inputs at natural frequency resonate, i.e., they are amplified
- Natural frequencies should be high enough to minimize likelihood of resonant response

Response to Oscillatory Input

\[
\begin{align*}
[A \omega^2 \sin(\omega t + \pi)] + 2\xi \omega_n [A \omega \sin(\omega t + \pi/2)] + \omega_n^2 [A \sin(\omega t)] &= \omega_n^2 [B \sin(\omega t)]
\end{align*}
\]

- ... however, A must be a complex number for this to work
- A better way: Compute Laplace transform to find transfer functions

\[
L[\Delta x(t)] = \Delta x(s) = \int_0^t \Delta x(t)e^{-st} dt, \quad s = \sigma + j\omega, \quad (j = i = \sqrt{-1})
\]

- Neglecting initial conditions

\[
\begin{align*}
L[\Delta \dot{x}(t)] &= s\Delta x(s) \\
L[\Delta \ddot{x}(t)] &= s^2\Delta x(s)
\end{align*}
\]
**Transfer Function**

\[ L(\Delta \ddot{x} + 2\zeta \omega_n \Delta \dot{x} + \omega_n^2 \Delta x) = L(\omega_n^2 \Delta u) \]

or

\[ (s^2 + 2\zeta \omega_n s + \omega_n^2) \Delta x(s) = \omega_n^2 \Delta u(s) \]

Transfer function from input to displacement

\[
\frac{\Delta x(s)}{\Delta u(s)} = \frac{\omega_n^2}{(s^2 + 2\zeta \omega_n s + \omega_n^2)}
\]

**Transfer Functions of Displacement, Velocity, and Acceleration**

- Transfer function from input to displacement

  \[
  \frac{\Delta x(s)}{\Delta u(s)} = \frac{\omega_n^2}{(s^2 + 2\zeta \omega_n s + \omega_n^2)}
  \]

- Input to velocity: multiply by \(s\)

  \[
  \frac{\Delta \dot{x}(s)}{\Delta u(s)} = \frac{\omega_n^2 s}{(s^2 + 2\zeta \omega_n s + \omega_n^2)}
  \]

- Input to acceleration: multiply by \(s^2\)

  \[
  \frac{\Delta \ddot{x}(s)}{\Delta u(s)} = \frac{\omega_n^2 s^2}{(s^2 + 2\zeta \omega_n s + \omega_n^2)}
  \]
From Transfer Function to Frequency Response

- Displacement transfer function

\[
\frac{\Delta x(s)}{\Delta u(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}
\]

- Displacement frequency response \((s = j\omega)\)

\[
\frac{\Delta x(j\omega)}{\Delta u(j\omega)} = \frac{\omega_n^2}{(j\omega)^2 + 2\zeta\omega_n(j\omega) + \omega_n^2}
\]

Frequency Response

- \(\omega_n\): natural frequency of the system
- \(\omega\): frequency of a sinusoidal input to the system

\[
\frac{\Delta x(j\omega)}{\Delta u(j\omega)} = \frac{\omega_n^2}{(j\omega)^2 + 2\zeta\omega_n(j\omega) + \omega_n^2} = \frac{\omega_n^2}{\omega_n^2 - \omega^2 + 2\zeta\omega_n(j\omega) + \omega_n^2} = \frac{\omega_n^2}{c(\omega) + j\omega_n(j\omega)} = \frac{c(\omega) - j\omega_n(j\omega)}{c(\omega) + j\omega_n(j\omega)} \omega_n^2 = \frac{c(\omega) - j\omega_n(j\omega)}{c(\omega) + j\omega_n(j\omega)} = \frac{c(\omega) - j\omega_n(j\omega)}{c(\omega) + d^2(\omega)}
\]

\[\equiv a(\omega) + jb(\omega) \equiv A(\omega)e^{j\phi(\omega)}\]

- Frequency response is a complex function
  - Real and imaginary components, or
  - Amplitude and phase angle
Bending Vibrations of a Free-Free Uniform Beam

\[ EI_y \frac{d^4 z}{dx^4} \bigg|_{x=x_s} = k = -m' \frac{d^2 z}{dt^2} \bigg|_{x=x_s} \]

- \( EI_y \) = constant
- \( m' \) = mass variation with length (constant)
- \( k \) = effective spring constant

- Solution by separation of variables requires that left and right sides equal a constant, \( k \)
- An infinite number of separation constants, \( k_i \), exist
- Therefore, there are an infinite number of vibrational response modes

Mode shapes of bending vibrations

4th mode
2nd mode
1st mode
3rd mode

\( u_i (y) \)
Vibrational Mode Shapes for Saturn 5

Computational Grid Model

Shapes of the First Seven Modes

- Body elastic deflection distorts the shape of scramjet inlet and exhaust ramps
- Aeroelastic-propulsive interactions
- Impact on flight dynamics
Next Time:
Chariots for Apollo
Spacecraft Design

Supplemental Material