Aircraft Equations of Motion: Translation and Rotation

Robert Stengel, Aircraft Flight Dynamics, MAE 331, 2018

Learning Objectives

- What use are the equations of motion?
- How is the angular orientation of the airplane described?
- What is a cross-product-equivalent matrix?
- What is angular momentum?
- How are the inertial properties of the airplane described?
- How is the rate of change of angular momentum calculated?

Reading:
Flight Dynamics
155–161

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http://www.princeton.edu/~stengel/MAE331.html
http://www.princeton.edu/~stengel/FlightDynamics.html

Review Questions

- What characteristic(s) provide maximum gliding range?
- Do gliding heavy airplanes fall out of the sky faster than light airplanes?
- Are the factors for maximum gliding range and minimum sink rate the same?
- How does the maximum climb rate vary with altitude?
- What are “energy height” and “specific excess power”?
- What is an “energy climb”?
- How is the “maneuvering envelope” defined?
- What factors determine the maximum steady turning rate?
**Dynamic Systems**

**Dynamic Process:** Current state depends on prior state
- \( x = \text{dynamic state} \)
- \( u = \text{input} \)
- \( w = \text{exogenous disturbance} \)
- \( p = \text{parameter} \)
- \( t \text{ or } k = \text{time or event index} \)

\[
\frac{dx(t)}{dt} = f[x(t), u(t), w(t), p(t), t]
\]

**Observation Process:** Measurement may contain error or be incomplete
- \( y = \text{output (error-free)} \)
- \( z = \text{measurement} \)
- \( n = \text{measurement error} \)

\[
y(t) = h[x(t), u(t)]
\]

\[
z(t) = y(t) + n(t)
\]

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**Ordinary Differential Equations Fall Into 4 Categories**

**Nonlinear, time-varying models (NTV)**

**Nonlinear, time-invariant models (NTI)**

**Linear, time-varying models (LTV)**

**Linear, time-invariant models (LTI)**

\[
\frac{dx(t)}{dt} = f[x(t), u(t), w(t)]
\]

\[
\frac{dx(t)}{dt} = F(t)x(t) + G(t)u(t) + L(t)w(t)
\]

\[
\frac{dx(t)}{dt} = Fx(t) + Gu(t) + Lw(t)
\]
What Use are the Equations of Motion?

- **Nonlinear equations of motion**
  - Compute “exact” flight paths and motions
    - Simulate flight motions
    - Optimize flight paths
    - Predict performance
  - Provide basis for approximate solutions

\[
\frac{dx(t)}{dt} = f[x(t), u(t), w(t), p(t), t]
\]

\[
\frac{dx(t)}{dt} = Fx(t) + Gu(t) + Lw(t)
\]

- **Linear equations of motion**
  - Simplify computation of flight paths and solutions
  - Define modes of motion
  - Provide basis for control system design and flying qualities analysis

Examples of Airplane Dynamic System Models

- **Nonlinear, Time-Varying**
  - Large amplitude motions
  - Significant change in mass

- **Linear, Time-Varying**
  - Small amplitude motions
  - Perturbations from a dynamic flight path

- **Nonlinear, Time-Invariant**
  - Large amplitude motions
  - Negligible change in mass

- **Linear, Time-Invariant**
  - Small amplitude motions
  - Perturbations from an equilibrium flight path
Translational Position

Position of a Particle

Projections of vector magnitude on three axes

\[
\mathbf{r} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = r \begin{bmatrix} \cos \delta_x \\ \cos \delta_y \\ \cos \delta_z \end{bmatrix}
\]

\[
\begin{bmatrix} \cos \delta_x \\ \cos \delta_y \\ \cos \delta_z \end{bmatrix} = \text{Direction cosines}
\]
Cartesian Frames of Reference

- Two reference frames of interest
  - \( I \): Inertial frame (fixed to inertial space)
  - \( B \): Body frame (fixed to body)

- Translation
  - Relative linear positions of origins

- Rotation
  - Orientation of the body frame with respect to the inertial frame

Common convention (\( z \) up)  
Aircraft convention (\( z \) down)

Measurement of Position in Alternative Frames - 1

- Two reference frames of interest
  - \( I \): Inertial frame (fixed to inertial space)
  - \( B \): Body frame (fixed to body)

\[
\mathbf{r} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}
\]

\[
\mathbf{r}_{\text{particle}} = \mathbf{r}_{\text{origin}} + \Delta \mathbf{r}_{\text{inertia to body}}
\]

- Differences in frame orientations must be taken into account in adding vector components

Inertial-axis view  
Body-axis view
Measurement of Position in Alternative Frames - 2

Inertial-axis view

\[ \mathbf{r}_{\text{particle}_i} = \mathbf{r}_{\text{origin-}B_i} + \mathbf{H}_B^I \Delta \mathbf{r}_B \]

Body-axis view

\[ \mathbf{r}_{\text{particle}_B} = \mathbf{r}_{\text{origin-}I_B} + \mathbf{H}_I^B \Delta \mathbf{r}_I \]

Rotational Orientation
Direction Cosine Matrix

\[ H_I^B = \begin{bmatrix}
\cos \delta_{11} & \cos \delta_{21} & \cos \delta_{31} \\
\cos \delta_{12} & \cos \delta_{22} & \cos \delta_{32} \\
\cos \delta_{13} & \cos \delta_{23} & \cos \delta_{33}
\end{bmatrix} \]

- Projections of unit vector components of one reference frame on another
- Rotational orientation of one reference frame with respect to another
- Cosines of angles between each \( I \) axis and each \( B \) axis

\[ r_B = H_I^B r_I \]

Properties of the Rotation Matrix

\[ H_I^B = \begin{bmatrix}
\cos \delta_{11} & \cos \delta_{21} & \cos \delta_{31} \\
\cos \delta_{12} & \cos \delta_{22} & \cos \delta_{32} \\
\cos \delta_{13} & \cos \delta_{23} & \cos \delta_{33}
\end{bmatrix}_I = \begin{bmatrix}
h_{11} & h_{12} & h_{13} \\
h_{21} & h_{22} & h_{23} \\
h_{31} & h_{32} & h_{33}
\end{bmatrix}_I \]

\[ r_B = H_I^B r_I \quad s_B = H_I^B s_I \]

Orthonormal transformation

Angles between vectors are preserved
Lengths are preserved

\[ |r_I| = |r_B| \quad |s_I| = |s_B| \]
\[ \angle(r_I, s_I) = \angle(r_B, s_B) = \text{x deg} \]
Euler Angles

- Body attitude measured with respect to inertial frame
- Three-angle orientation expressed by sequence of three orthogonal single-angle rotations

\[ \text{Inertial} \Rightarrow \text{Intermediate}_1 \Rightarrow \text{Intermediate}_2 \Rightarrow \text{Body} \]

- 24 (±12) possible sequences of single-axis rotations
- Aircraft convention: 3-2-1, z positive down

\[ \psi \]: Yaw angle  
\[ \theta \]: Pitch angle  
\[ \phi \]: Roll angle

Euler Angles Measure the Orientation of One Frame with Respect to the Other

- Conventional sequence of rotations from inertial to body frame
  - Each rotation is about a single axis
  - Right-hand rule
  - Yaw, then pitch, then roll
  - These are called Euler Angles

Other sequences of 3 rotations can be chosen; however, once sequence is chosen, it must be retained
**Reference Frame Rotation from Inertial to Body: Aircraft Convention (3-2-1)**

**Yaw rotation (ψ) about z axis**

\[
\begin{bmatrix}
  x' \\ y' \\ z'
\end{bmatrix} =
\begin{bmatrix}
  \cos \psi & \sin \psi & 0 \\
  -\sin \psi & \cos \psi & 0 \\
  0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
  x \\ y \\ z
\end{bmatrix}
\]

\[r_z = H_z^i r\]

**Pitch rotation (θ) about y axis**

\[
\begin{bmatrix}
  x' \\ y' \\ z'
\end{bmatrix} =
\begin{bmatrix}
  \cos \theta & 0 & -\sin \theta \\
  0 & 1 & 0 \\
  \sin \theta & 0 & \cos \theta
\end{bmatrix}
\begin{bmatrix}
  x \\ y \\ z
\end{bmatrix}
\]

\[r_y = H_y^i r\]

**Roll rotation (φ) about x axis**

\[
\begin{bmatrix}
  x' \\ y' \\ z'
\end{bmatrix} =
\begin{bmatrix}
  1 & 0 & 0 \\
  0 & \cos \phi & \sin \phi \\
  0 & -\sin \phi & \cos \phi
\end{bmatrix}
\begin{bmatrix}
  x \\ y \\ z
\end{bmatrix}
\]

\[r_x = H_x^i r\]

---

**The Rotation Matrix**

The three-angle rotation matrix is the product of 3 single-angle rotation matrices:

\[
H^B_i(\phi, \theta, \psi) = H^B_x(\phi) H^B_y(\theta) H^B_z(\psi)
\]

\[
\begin{bmatrix}
  1 & 0 & 0 \\
  0 & \cos \phi & \sin \phi \\
  0 & -\sin \phi & \cos \phi
\end{bmatrix}
\begin{bmatrix}
  \cos \theta & 0 & -\sin \theta \\
  0 & 1 & 0 \\
  \sin \theta & 0 & \cos \theta
\end{bmatrix}
\begin{bmatrix}
  \cos \psi & \sin \psi & 0 \\
  -\sin \psi & \cos \psi & 0 \\
  0 & 0 & 1
\end{bmatrix}
\]

an expression of the Direction Cosine Matrix
Rotation Matrix Inverse
Inverse relationship: interchange sub- and superscripts

\[ r_B = H_I^B r_I \]
\[ r_I = (H_I^B)^{-1} r_B = H_I^B r_B \]

Because transformation is **orthonormal**
Inverse = transpose
Rotation matrix is always **non-singular**

\[ [H_I^B(\phi, \theta, \psi)]^{-1} = [H_I^B(\phi, \theta, \psi)]^T = H_I^B(\psi, \theta, \phi) \]
\[ H_B^I = (H_I^B)^{-1} = (H_I^B)^T = H_I^1 H_I^2 H_B^2 \]
\[ H_B^I H_I^B = H_I^B H_B^I = I \]

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**Checklist**

- What are direction cosines?
- What are Euler angles?
- What rotation sequence is used to describe airplane attitude?
- What are properties of the rotation matrix?
Angular Momentum

Angular Momentum of a Particle

- Moment of linear momentum of differential particles that make up the body
  - (Differential masses) x components of the velocity that are perpendicular to the moment arms

\[
d\mathbf{L} = \mathbf{r} \times dm \mathbf{v} = (\mathbf{r} \times \mathbf{v}_m)dm = \left[ \mathbf{r} \times (\mathbf{v}_o + \mathbf{\omega} \times \mathbf{r}) \right]dm
\]

- Cross Product: Evaluation of a determinant with unit vectors \((i, j, k)\) along axes, \((x, y, z)\) and \((v_x, v_y, v_z)\) projections on to axes

\[
\mathbf{r} \times \mathbf{v} = \begin{vmatrix}
  i & j & k \\
  x & y & z \\
  v_x & v_y & v_z
\end{vmatrix} = (yv_z - vz_y)i + (zv_x - xv_z)j + (xv_y - yv_x)k
\]
Cross-Product-Equivalent Matrix

\[ \mathbf{r} \times \mathbf{v} = \begin{vmatrix} i & j & k \\ x & y & z \\ v_x & v_y & v_z \end{vmatrix} = (yv_z - zv_y)i + (zv_x - xv_z)j + (xv_y - yv_x)k \]

\[ = \begin{bmatrix} (yv_z - zv_y) \\ (zv_x - xv_z) \\ (xv_y - yv_x) \end{bmatrix} = \mathbf{\tilde{r}} \mathbf{v} = \begin{bmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{bmatrix} \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix} \]

Cross-product-equivalent matrix

\[ \mathbf{\tilde{r}} = \begin{bmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{bmatrix} \]

Angular Momentum of the Aircraft

- Integrate moment of linear momentum of differential particles over the body

\[ \mathbf{h} = \int_{\text{Body}} \left[ \mathbf{r} \times (\mathbf{v} + \mathbf{\omega} \times \mathbf{r}) \right] \, d\mathbf{m} = \int_{x_{\text{min}}}^{x_{\text{max}}} \int_{y_{\text{min}}}^{y_{\text{max}}} \int_{z_{\text{min}}}^{z_{\text{max}}} \rho(x,y,z) \mathbf{r} \times \mathbf{v} \, dx \, dy \, dz = \begin{bmatrix} h_x \\ h_y \\ h_z \end{bmatrix} \]

\[ \rho(x,y,z) = \text{Density of the body} \]

- Choose the center of mass as the rotational center

\[ \mathbf{h} = \int_{\text{Body}} (\mathbf{r} \times \mathbf{v}) \, d\mathbf{m} + \int_{\text{Body}} (\mathbf{r} \times (\mathbf{\omega} \times \mathbf{r})) \, d\mathbf{m} = 0 - \int_{\text{Body}} (\mathbf{r} \times (\mathbf{r} \times \mathbf{\omega})) \, d\mathbf{m} = -\int_{\text{Body}} (\mathbf{r} \times \mathbf{r}) \, d\mathbf{m} \times \mathbf{\omega} \equiv -\int_{\text{Body}} (\mathbf{\tilde{r}} \mathbf{r}) \, d\mathbf{m} \mathbf{\omega} \]

Supermarine Spitfire

Center of Mass
Location of the Center of Mass

\[ \mathbf{r}_{cm} = \frac{1}{m} \int_{Body} \mathbf{r} \, dm = \int_{x_{min}}^{x_{max}} \int_{y_{min}}^{y_{max}} \int_{z_{min}}^{z_{max}} \mathbf{r} \rho(x, y, z) \, dx \, dy \, dz \]

\[
\begin{bmatrix}
\mathbf{x}_{cm} \\
\mathbf{y}_{cm} \\
\mathbf{z}_{cm}
\end{bmatrix}
\]

The Inertia Matrix
The Inertia Matrix

\[ h = - \int \ddot{r} r \, \omega \, dm = - \int \ddot{r} r \, dm \, \omega = I \omega \]

\[ \omega = \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} \]

\[ I = - \int \ddot{r} \, dm = - \int \begin{bmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{bmatrix} \begin{bmatrix} -z & y \\ z & 0 & -x \\ -y & x & 0 \end{bmatrix} \, dm \]

\[ = \int \begin{bmatrix} (y^2 + z^2) & -xy & -xz \\ -xy & (x^2 + z^2) & -yz \\ -xz & -yz & (x^2 + y^2) \end{bmatrix} \, dm \]

Inertia matrix derives from equal effect of angular rate on all particles of the aircraft

Moments and Products of Inertia

\[ I = \int \begin{bmatrix} (y^2 + z^2) & -xy & -xz \\ -xy & (x^2 + z^2) & -yz \\ -xz & -yz & (x^2 + y^2) \end{bmatrix} \, dm = \begin{bmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{yx} & I_{yy} & -I_{yz} \\ -I_{zx} & -I_{zy} & I_{zz} \end{bmatrix} \]

Inertia matrix

- Moments of inertia on the diagonal
- Products of inertia off the diagonal
- If products of inertia are zero, \((x, y, z)\) are principal axes --->
- All rigid bodies have a set of principal axes

Ellipsoid of Inertia

\[ I_{xx}x^2 + I_{yy}y^2 + I_{zz}z^2 = 1 \]
Inertia Matrix of an Aircraft with Mirror Symmetry

\[ I = \int_{\text{Body}} \begin{bmatrix} (y^2 + z^2) & 0 & -xz \\ 0 & (x^2 + z^2) & 0 \\ -xz & 0 & (x^2 + y^2) \end{bmatrix} \, dm = \begin{bmatrix} I_{xx} & 0 & -I_{xz} \\ 0 & I_{yy} & 0 \\ -I_{xz} & 0 & I_{zz} \end{bmatrix} \]

Nose high/low product of inertia, \( I_{xz} \)

Nominal Configuration
Tips folded, 50% fuel, \( W = 38,524 \) lb
\[ x_m @ 0.218 \, \text{r} \]
\[ I_{xx} = 1.8 \times 10^6 \, \text{slug-ft}^2 \]
\[ I_{yy} = 19.9 \times 10^6 \, \text{slug-ft}^2 \]
\[ I_{zz} = 22.1 \times 10^6 \, \text{slug-ft}^2 \]
\[ I_{xz} = -0.88 \times 10^6 \, \text{slug-ft}^2 \]

Checklist
- How is the location of the center of mass found?
- What is a cross-product-equivalent matrix?
- What is the inertia matrix?
- What is an ellipsoid of inertia?
- What does the "nose-high" product of inertia represent?
**Historical Factoids**

**Technology of World War II Aviation**

- **1938-45**: Analytical and experimental approach to design
  - Many configurations designed and flight-tested
  - Increased specialization; radar, navigation, and communication
  - Approaching the “sonic barrier”

- **Aircraft Design**
  - Large, powerful, high-flying aircraft
  - Turbocharged engines
  - Oxygen and Pressurization

**Power Effects on Stability and Control**

- **Brewster Buffalo**: over-armored and under-powered
- During W.W.II, the size of fighters remained about the same, but installed horsepower doubled (**F4F** vs. **F8F**)
- Use of flaps means high power at low speed, increasing relative significance of thrust effects
**World War II Carrier-Based Airplanes**

- Takeoff without catapult, relatively low landing speed [http://www.youtube.com/watch?v=4dySbhKTVNk](http://www.youtube.com/watch?v=4dySbhKTVNk)
- Tailhook and arresting gear
- Carrier steams into wind
- Design for storage (short tail length, folding wings) affects stability and control

**Multi-Engine Aircraft of World War II**

- Large W.W.II aircraft had unpowered controls:
  - High foot-pedal force
  - Rudder stability problems arising from balancing to reduce pedal force
- Severe engine-out problem for twin-engine aircraft
**WW II Military Flying Boats**

Seaplanes proved useful during World War II

- Martin PB2M Mars
- Lockheed PBY Catalina
- Martin PB2N Narwhal
- Boeing XPBB Sea Ranger
- Saunders-Roe SR.36 Lerwick
- Grumman JRF-1 Goose

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**Rate of Change of Angular Momentum**
Newton’s 2nd Law, Applied to Rotational Motion

In inertial frame, rate of change of angular momentum = applied moment (or torque), $M$

$$\frac{d\mathbf{h}}{dt} = \frac{d(\mathbf{I}\omega)}{dt} = \frac{d\mathbf{I}}{dt}\omega + \mathbf{I}\frac{d\omega}{dt} = \mathbf{I}\dot{\omega} + \dot{\mathbf{I}}\omega = M = \begin{bmatrix} m_x \\ m_y \\ m_z \end{bmatrix}$$

Angular Momentum and Rate

Angular momentum and rate vectors are not necessarily aligned

$$\mathbf{h} = \mathbf{I}\omega$$
How Do We Get Rid of \( \frac{dI}{dt} \) in the Angular Momentum Equation?

Chain Rule \[ \frac{d(I\omega)}{dt} = \dot{I}\omega + I\dot{\omega} \] \[ \dot{\bar{I}} \neq 0 \]

- **Dynamic equation in a body-referenced frame**
  - Inertial properties of a constant-mass, rigid body are **unchanging** in a body frame of reference
  - ... **but** a body-referenced frame is “non-Newtonian” or “non-inertial”
  - Therefore, dynamic equation must be **modified** for expression in a rotating frame

Angular Momentum Expressed in Two Frames of Reference

- **Angular momentum and rate** are vectors
  - Expressed in either the **inertial** or **body frame**
  - Two frames related algebraically by the **rotation matrix**

\[
\begin{align*}
\mathbf{h}_B(t) &= \mathbf{H}_I^B(t) \mathbf{h}_I(t); & \mathbf{h}_I(t) &= \mathbf{H}_B^I(t) \mathbf{h}_B(t) \\
\mathbf{\omega}_B(t) &= \mathbf{H}_I^B(t) \mathbf{\omega}_I(t); & \mathbf{\omega}_I(t) &= \mathbf{H}_B^I(t) \mathbf{\omega}_B(t)
\end{align*}
\]
Vector Derivative Expressed in a Rotating Frame

Chain Rule
\[ \dot{\mathbf{h}}_I = \mathbf{H}_B^T \dot{\mathbf{h}}_B + \dot{\mathbf{H}}_B^T \mathbf{h}_B \]

Effect of body-frame rotation

Consequently, the 2nd term is
\[ \dot{\mathbf{H}}_B^T \mathbf{h}_B = \tilde{\omega}_I \mathbf{h}_I = \mathbf{H}_B^T \dot{\mathbf{h}}_B + \tilde{\omega}_I \mathbf{h}_I \]

Rate of change expressed in body frame

Alternatively
\[ \dot{\mathbf{h}}_I = \mathbf{H}_B^T \dot{\mathbf{h}}_B + \omega_I \times \mathbf{h}_I = \mathbf{H}_B^T \dot{\mathbf{h}}_B + \tilde{\omega}_I \mathbf{h}_I \]

Consequently, the 2nd term is
\[ \dot{\mathbf{H}}_B^T \mathbf{h}_B = \tilde{\omega}_I \mathbf{h}_I = \mathbf{H}_B^T \dot{\mathbf{h}}_B + \tilde{\omega}_I \mathbf{h}_I \]

... where the cross-product equivalent matrix of angular rate is
\[ \tilde{\omega} = \begin{bmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{bmatrix} \]

External Moment Causes Change in Angular Rate

Positive rotation of Frame B w.r.t. Frame A is a negative rotation of Frame A w.r.t. Frame B

In the body frame of reference, the angular momentum change is
\[ \mathbf{h}_B = \mathbf{H}_B^T \mathbf{h}_I + \mathbf{H}_B^T \dot{\mathbf{h}}_I = \mathbf{H}_B^T \mathbf{h}_I - \omega_B \times \mathbf{h}_B = \mathbf{H}_B^T \mathbf{h}_I - \tilde{\omega}_B \mathbf{h}_B \]
\[ = \mathbf{H}_B^T \mathbf{M}_I - \tilde{\omega}_B \mathbf{I}_B \omega_B = \mathbf{M}_B - \tilde{\omega}_B \mathbf{I}_B \omega_B \]

Moment = torque = force x moment arm
\[ \mathbf{M}_I = \begin{bmatrix} m_x \\ m_y \\ m_z \end{bmatrix}; \quad \mathbf{M}_B = \mathbf{H}_B^T \mathbf{M}_I = \begin{bmatrix} m_x \\ m_y \\ m_z \end{bmatrix} = \begin{bmatrix} L \\ M \\ N \end{bmatrix} \]
Rate of Change of Body-Referenced Angular Rate due to External Moment

In the body frame of reference, the angular momentum change is

\[
\dot{\mathbf{h}}_B = \mathbf{H}^B_{\omega} \dot{\mathbf{h}}_B + \mathbf{H}^B_{\mathbf{M}} = \mathbf{H}^B_{\omega} \dot{\mathbf{h}}_B - \omega_B \times \mathbf{h}_B
\]

\[
= \mathbf{H}^B_{\omega} \dot{\mathbf{h}}_B - \mathbf{\bar{\omega}}_B / h_B = \mathbf{H}^B_{\omega} \mathbf{I}_B - \mathbf{\bar{\omega}}_B \mathbf{I}_B \omega_B
\]

\[
= \mathbf{M}_B - \mathbf{\bar{\omega}}_B \mathbf{I}_B \omega_B
\]

For constant body-axis inertia matrix

\[
\dot{\mathbf{h}}_B = \mathbf{I}_B \dot{\omega}_B = \mathbf{M}_B - \mathbf{\bar{\omega}}_B \mathbf{I}_B \omega_B
\]

Consequently, the differential equation for angular rate of change is

\[
\dot{\omega}_B = \mathbf{I}^{-1}_B \left( \mathbf{M}_B - \mathbf{\bar{\omega}}_B \mathbf{I}_B \omega_B \right)
\]

Checklist

- Why is it inconvenient to solve momentum rate equations in an inertial reference frame?
- Are angular rate and momentum vectors aligned?
- How are angular rate equations transformed from an inertial to a body frame?
Next Time:
Aircraft Equations of Motion:
Flight Path Computation

Reading:
Flight Dynamics
161-180

Learning Objectives

How is a rotating reference frame described in an inertial reference frame?
Is the transformation singular?
What adjustments must be made to expressions for forces and moments in a non-inertial frame?
How are the 6-DOF equations implemented in a computer?
Damping effects
Moments and Products of Inertia

(Bedford & Fowler)

Moments and products of inertia tabulated for geometric shapes with uniform density

Construct aircraft moments and products of inertia from components using parallel-axis theorem

Model in CREO, etc.

\[
I_{xx} = I_{yy} = \frac{1}{2} mR^2, \\
I_{zz} = I_{xy} = I_{yz} = I_{zx} = 0.
\]

Thin circular plate

\[
I_{xx} = I_{yy} = I_{xy} = I_{yx} = I_{yz} = I_{zy} = I_{zx} = I_{xz} = 0.
\]

Slender bar

\[
I_{xx} = \frac{1}{12} m(a^2 + b^2), \\
I_{yy} = \frac{1}{12} m(a^2 + c^2), \\
I_{zz} = \frac{1}{12} m(b^2 + c^2), \\
I_{xy} = I_{yx} = I_{xz} = I_{zx} = 0.
\]

Rectangular prism

\[
I_{xx} = \frac{1}{12} m(a^4 + b^4) - \frac{1}{4} mR^2, \\
I_{yy} = \frac{1}{12} m(a^4 + c^4) - \frac{1}{4} mR^2, \\
I_{zz} = \frac{1}{12} m(b^4 + c^4) - \frac{1}{4} mR^2, \\
I_{xy} = I_{yx} = I_{xz} = I_{zx} = 0.
\]

Circular cylinder