Point-Mass Dynamics and Aerodynamic/Thrust Forces
Robert Stengel, Aircraft Flight Dynamics, MAE 331, 2018

Learning Objectives
- Properties of atmosphere
- Frames of reference
- Velocity and momentum
- Newton’s laws of motion
- Airplane axes
- Lift and drag
- Simplified equations for longitudinal motion
- Powerplants and thrust

The Atmosphere
Properties of the Lower Atmosphere*

- Density and pressure decay exponentially with altitude
- Temperature and speed of sound are piecewise-linear functions of altitude

* 1976 US Standard Atmosphere

Air Density, Dynamic Pressure, and Mach Number

\[
\rho = \text{Air density, function of height} = \rho_{\text{sealevel}} e^{-\beta h} = \rho_{\text{sealevel}} e^{\beta z}
\]

\[
\rho_{\text{sealevel}} = 1.225 \text{ kg/m}^3, \quad \beta = 1/9,042 \text{ m}
\]

\[
V_{\text{air}} = \left[ v_x^2 + v_y^2 + v_z^2 \right]^{1/2} = \left[ \mathbf{v}^T \mathbf{v} \right]^{1/2} = \text{Airspeed}
\]

Dynamic pressure \( \bar{q} = \frac{1}{2} \rho(h) V_{\text{air}}^2 \triangleq \text{"qbar"} \)

Mach number \( M = \frac{V_{\text{air}}}{a(h)} \); \( a = \text{speed of sound, m/s} \)
Contours of Constant Dynamic Pressure, $\overline{q}$

- In steady, cruising flight, \[ \text{Weight} = \text{Lift} = C_L \frac{1}{2} \rho V^2 \overline{q} \]

\begin{align*}
\text{True airspeed} & \text{ must increase as altitude increases} \\
& \text{to maintain constant dynamic pressure}
\end{align*}

Wind: Motion of the Atmosphere

Zero wind at Earth’s surface = Rotating air mass

Wind measured with respect to Earth’s rotating surface

Wind Velocity Profiles vary over Time

Airspeed = Airplane’s speed with respect to air mass

Earth-relative velocity = Wind velocity $\pm$ True airspeed $[\text{vector}]$
Historical Factoids

- **Henri Pitot**: Pitot tube (1732)
- **Benjamin Robins**: Whirling arm "wind tunnel" (1742)
- **Sir George Cayley**
  - Sketched "modern" airplane configuration (1799)
  - Hand-launched glider (1804)
  - Applied aerodynamics (1809-1810)
  - Triplane glider carrying 10-yr-old boy (1849)
  - Monoplane glider carrying coachman (1853)
    - Cayley's coachman had a steering oar with cruciform blades

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Equations of Motion for a Particle (Point Mass)
Newtonian Frame of Reference

- Newtonian (Inertial) Reference Frame
  - Unaccelerated Cartesian frame: origin referenced to inertial (non-moving) frame
  - Right-hand rule
  - Origin can translate at constant linear velocity
  - Frame cannot rotate with respect to inertial origin
- Position: 3 dimensions
- What is a non-moving frame?

Translation changes the position of an object

Velocity and Momentum

- Velocity of a particle
  \[
  \mathbf{v} = \frac{d\mathbf{X}}{dt} = \dot{\mathbf{X}} = \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix}
  \]

- Linear momentum of a particle
  \[
  \mathbf{p} = m\mathbf{v} = m
  \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix}
  \]
  where \( m = \text{mass of particle} \)
Inertial Velocity Expressed in Polar Coordinates

Polar Coordinates

Projected on a Sphere

\( \gamma \): Vertical Flight Path Angle, rad or deg

\( \xi \): Horizontal Flight Path Angle (Heading Angle), rad or deg

Newton’s Laws of Motion: Dynamics of a Particle

- **First Law**: If no force acts on a particle,
  - it remains at rest or
  - continues to move in a straight line at constant velocity, as observed in an inertial reference frame
- **Momentum is conserved**

\[
\frac{d}{dt}(mv) = 0 ; \quad mv|_{t_1} = mv|_{t_2}
\]
Newton’s Laws of Motion: Dynamics of a Particle

- **Second Law**: Particle of fixed mass acted upon by a force
  - changes velocity with acceleration proportional to and in direction of force, as observed in inertial frame
  - Mass is ratio of force to acceleration of particle:

\[
\frac{d}{dt}(mv) = m\frac{dv}{dt} = F; \quad F = \begin{bmatrix} f_x \\ f_y \\ f_z \end{bmatrix}
\]

\[
\therefore \frac{dv}{dt} = \frac{1}{m}F = \frac{1}{m} \begin{bmatrix} 1/m & 0 & 0 \\ 0 & 1/m & 0 \\ 0 & 0 & 1/m \end{bmatrix} \begin{bmatrix} f_x \\ f_y \\ f_z \end{bmatrix}
\]

Newton’s Laws of Motion: Dynamics of a Particle

- **Third Law**
  - For every action, there is an equal and opposite reaction

**Force on Rocket** = **Force on Exhaust Gasses**
Equations of Motion for a Particle: Position and Velocity

\[ \mathbf{F}_x = \begin{bmatrix} f_x \\ f_y \\ f_z \end{bmatrix} = \mathbf{F}_{\text{gravity}} + \mathbf{F}_{\text{aerodynamics}} + \mathbf{F}_{\text{drag}} \]

Rate of change of velocity

\[ \frac{d\mathbf{v}}{dt} = \mathbf{\dot{v}} = \begin{bmatrix} \dot{v}_x \\ \dot{v}_y \\ \dot{v}_z \end{bmatrix} = \frac{1}{m} \mathbf{F} = \frac{1}{m} \begin{bmatrix} f_x \\ f_y \\ f_z \end{bmatrix} \]

Rate of change of position

\[ \frac{d\mathbf{r}}{dt} = \mathbf{\dot{r}} = \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} = \mathbf{v} = \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix} \]

Integration for Velocity with Constant Force

Differential equation

\[ \frac{d\mathbf{v}(t)}{dt} = \mathbf{\dot{v}}(t) = \frac{1}{m} \mathbf{F} = \frac{1}{m} \begin{bmatrix} f_x/m \\ f_y/m \\ f_z/m \end{bmatrix} = \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix} \]

Integral

\[ \mathbf{v}(T) = \int_0^T \frac{d\mathbf{v}(t)}{dt} dt + \mathbf{v}(0) = \int_0^T \frac{1}{m} \mathbf{F} dt + \mathbf{v}(0) = \int_0^T \mathbf{a} dt + \mathbf{v}(0) \]

\[ \begin{bmatrix} v_x(T) \\ v_y(T) \\ v_z(T) \end{bmatrix} = \int_0^T \begin{bmatrix} f_x/m \\ f_y/m \\ f_z/m \end{bmatrix} dt + \begin{bmatrix} v_x(0) \\ v_y(0) \\ v_z(0) \end{bmatrix} = \int_0^T \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix} dt + \begin{bmatrix} v_x(0) \\ v_y(0) \\ v_z(0) \end{bmatrix} \]
Integration for Position with Varying Velocity

\[
\frac{d\mathbf{r}(t)}{dt} = \mathbf{v}(t) = \begin{bmatrix}
\dot{x}(t) \\
\dot{y}(t) \\
\dot{z}(t)
\end{bmatrix} = \begin{bmatrix}
v_x(t) \\
v_y(t) \\
v_z(t)
\end{bmatrix}
\]

\[
\mathbf{r}(T) = \int_0^T \frac{d\mathbf{r}(t)}{dt} \, dt + \mathbf{r}(0) = \int_0^T \mathbf{v}(t) \, dt + \mathbf{r}(0)
\]

\[
\begin{bmatrix}
x(T) \\
y(T) \\
z(T)
\end{bmatrix} = \int_0^T \begin{bmatrix}
v_x(t) \\
v_y(t) \\
v_z(t)
\end{bmatrix} \, dt + \begin{bmatrix}
x(0) \\
y(0) \\
z(0)
\end{bmatrix}
\]

Gravitational Force: Flat-Earth Approximation

- Approximation
  - Flat earth reference assumed to be inertial frame, e.g.,
    - North, East, Down
    - Range, Crossrange, Altitude (–)
- \( g \) is gravitational acceleration
- \( mg \) is gravitational force
- Independent of position
- \( z \) measured down

\[
\begin{bmatrix}
\mathbf{F}_{\text{gravity}}
\end{bmatrix}_I = \begin{bmatrix}
\mathbf{F}_{\text{gravity}}
\end{bmatrix}_E = m \begin{bmatrix}
0 \\
0 \\
g_o
\end{bmatrix}_E
\]

\( g_o \approx 9.807 \, \text{m/s}^2 \) at earth’s surface
Flight Path Dynamics, Constant Gravity, No Aerodynamics

\[ v_x(0) = v_{x_0} \]
\[ v_z(0) = v_{z_0} \]
\[ x(0) = x_0 \]
\[ z(0) = z_0 \]

\[ \dot{v}_x(t) = 0 \]
\[ \dot{v}_z(t) = -g \quad (z \text{ positive up}) \]
\[ x(t) = v_x(t) \]
\[ z(t) = v_z(t) \]

\[ v_x(T) = v_{x_0} \]
\[ v_z(T) = v_{z_0} - \int_0^T g \, dt = v_{z_0} - gT \]
\[ x(T) = x_0 + v_{x_0} T \]
\[ z(T) = z_0 + v_{z_0} T - \int_0^T g t \, dt = z_0 + v_{z_0} T - gT^2/2 \]

Flight Path with Constant Gravity and No Aerodynamics
Aerodynamic Force on an Airplane

Earth-Reference Frame

\[
F_j = \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} C_X \\ C_Y \\ C_Z \end{bmatrix}_{j_E} \frac{1}{2} \rho V^2 S
\]

Body-Axis Frame

\[
F_b = \begin{bmatrix} C_X \\ C_Y \\ C_Z \end{bmatrix}_{j_B} \bar{q} S
\]

Wind-Axis Frame

\[
F_v = \begin{bmatrix} C_D \\ C_Y \\ C_L \end{bmatrix}_{j_B} \bar{q} S
\]

Referenced to the Earth, not the aircraft (e.g., North, East, Down)

Aligned with the aircraft axes (e.g., Nose, Wing Tip, Down)

Aligned with and perpendicular to the velocity vector (Forward, Sideward, Down)

Non-Dimensional Aerodynamic Coefficients

Body-Axis Frame

\[
\begin{bmatrix} C_X \\ C_Y \\ C_Z \end{bmatrix}_{j_B} = \text{axial force coefficient} \quad \text{side force coefficient} \quad \text{normal force coefficient}
\]

Wind-Axis Frame

\[
\begin{bmatrix} C_D \\ C_Y \\ C_L \end{bmatrix}_{j_B} = \text{drag coefficient} \quad \text{side force coefficient} \quad \text{lift coefficient}
\]

- Functions of flight condition, control settings, and disturbances, e.g., \( C_L = C_L(\delta, M, \delta E) \)
- Non-dimensional coefficients allow application of sub-scale model wind tunnel data to full-scale airplane
**Longitudinal Variables**

- $u(t)$: axial velocity
- $w(t)$: normal velocity
- $V(t)$: velocity magnitude
- $\alpha(t)$: angle of attack
- $\gamma(t)$: flight path angle
- $\theta(t)$: pitch angle

$\gamma = \theta - \alpha$ (with wingtips level)

- along vehicle centerline
- perpendicular to centerline
- along net direction of flight
- angle between centerline and direction of flight
- angle between direction of flight and local horizontal
- angle between centerline and local horizontal

**Lateral-Directional Variables**

- $\beta(t)$: sideslip angle
- $\psi(t)$: yaw angle
- $\xi(t)$: heading angle
- $\phi(t)$: roll angle

$\xi = \psi + \beta$ (with wingtips level)

- angle between centerline and direction of flight
- angle between centerline and local horizontal
- angle between direction of flight and compass reference (e.g., north)
- angle between true vertical and body $z$ axis
Introduction to Lift and Drag

Lift and Drag are Referenced to Velocity Vector

\[ \text{Lift} = C_L \frac{1}{2} \rho V_{air}^2 S = \left[ C_{L0} + \frac{\partial C_L}{\partial \alpha} \alpha \right] \frac{1}{2} \rho V_{air}^2 S \]

- Lift components sum to produce total lift
  - Perpendicular to velocity vector
  - Pressure differential between upper and lower surfaces

\[ \text{Drag} = C_D \frac{1}{2} \rho V_{air}^2 S = \left[ C_{D0} + \varepsilon C_L^2 \right] \frac{1}{2} \rho V_{air}^2 S \]

- Drag components sum to produce total drag
  - Parallel and opposed to velocity vector
  - Skin friction, pressure differentials
2-D Aerodynamic Lift

\[ \text{Lift} = C_L \, \frac{1}{2} \rho V^2 \, S = \left( C_{L_{\text{ind}}} + C_{L_{\text{ind,tip}}} + C_{L_{\text{ind,root}}} \right) \frac{1}{2} \rho V^2 \, S = \left( C_{L_{\text{ind}}} + \frac{\partial C_L}{\partial \alpha} \alpha \right) qS \]

- Upper/lower speed difference proportional to angle of attack
- Stagnation points at leading and trailing edges
  - Kutta condition: Aft stagnation point at sharp trailing edge

**Streamlines**
Instantaneous tangent to velocity vector

**Streaklines**
Locus of particles passing through points
Dye injected at fixed points

[Link](http://www.diam.unige.it/~irro/profilo_e.html)

3-D Lift

- Inward-Outward Flow
- Inward flow over upper surface
- Outward flow over lower surface
- Bound vorticity of wing produces tip vortices
2-D vs. 3-D Lift

Finite aspect ratio reduces lift slope

What is aspect ratio?

Aerodynamic Drag

\[ \text{Drag} = C_D \frac{1}{2} \rho V^2 S = \left( C_{D_v} + C_{D_s} + C_{D_l} \right) \frac{1}{2} \rho V^2 S = \left[ C_{D_0} + \frac{\varepsilon C_L}{2} \right] \bar{q} S \]

- **Drag components**
  - Parasite drag (friction, interference, base pressure differential)
  - Wave drag (shock-induced pressure differential)
  - Induced drag (drag due to lift generation)

- **In steady, subsonic flight**
  - Parasite (form) drag increases as \( V^2 \)
  - Induced drag (due to lift) proportional to \( 1/V^2 \)
  - Total drag minimized at one particular airspeed
2-D Equations of Motion with Aerodynamics and Thrust

2-D Equations of Motion for a Point Mass

- Motions restricted to vertical plane
- Inertial frame, wind = 0
- \( z \) positive down, flat-earth assumption
- Point-mass location coincides with aircraft’s center of mass

\[
\begin{bmatrix}
\dot{x} \\
\dot{z} \\
\dot{v}_x \\
\dot{v}_z \\
\end{bmatrix} =
\begin{bmatrix}
v_x \\
v_z \\
f_x/m \\
f_z/m \\
\end{bmatrix}
\begin{bmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\end{bmatrix}
\begin{bmatrix}
x \\
z \\
v_x \\
v_z \\
\end{bmatrix}
+
\begin{bmatrix}
0 \\
0 \\
1/m \\
0 \\
\end{bmatrix}
\begin{bmatrix}
0 & 0 \\
0 & 0 \\
1/m & 0 \\
0 & 1/m \\
\end{bmatrix}
\begin{bmatrix}
\frac{1}{2} \rho(z)(v_x^2 + v_z^2) \\
(C_T \cos \theta + C_x) \eta S \\
(C_T \sin \theta + C_z) \eta S + mg_x \\
\end{bmatrix}
\]
Transform Velocity from Cartesian to Polar Coordinates

Inertial axes -> wind axes and back

\[
\begin{bmatrix}
\dot{x} \\
\dot{z}
\end{bmatrix} = \begin{bmatrix}
v_x \\
v_z
\end{bmatrix} = \begin{bmatrix}
V \cos \gamma \\
-V \sin \gamma
\end{bmatrix} \Rightarrow \begin{bmatrix}
v \\
\gamma
\end{bmatrix} = \begin{bmatrix}
\sqrt{v_x^2 + v_z^2} \\
\sin^{-1} \left( \frac{v_z}{V} \right)
\end{bmatrix} = \begin{bmatrix}
\sqrt{v_x^2 + v_z^2} \\
\sin^{-1} \left( \frac{v_z}{V} \right)
\end{bmatrix}
\]

Rates of change of velocity and flight path angle

\[
\begin{bmatrix}
\dot{V} \\
\dot{\gamma}
\end{bmatrix} = \begin{bmatrix}
\frac{d}{dt} \sqrt{v_x^2 + v_z^2} \\
-\frac{d}{dt} \sin^{-1} \left( \frac{v_z}{V} \right)
\end{bmatrix}
\]

Longitudinal Point-Mass Equations of Motion

\[
\begin{align*}
\dot{x}(t) &= v_x = V(t) \cos \gamma(t) \\
\dot{z}(t) &= v_z = -V(t) \sin \gamma(t) \\
\dot{V}(t) &= \frac{\left( C_T \cos \alpha - C_D \right)}{2} \rho(z) V^2(t) S - m g_o \sin \gamma(t) \\
\dot{\gamma}(t) &= \frac{\left( C_T \sin \alpha + C_L \right)}{2} \rho(z) V^2(t) S - m g_o \cos \gamma(t)
\end{align*}
\]

x: range 
\(z\): height (altitude) 
\(V\): velocity 
\(\gamma\): flight path angle
Steady, Level (i.e., Cruising) Flight

In steady, level flight with $\cos \alpha \sim 1$, $\sin \alpha \sim 0$
Thrust = Drag, Lift = Weight

\[
V_{\text{cruise}} = \dot{x}(t) = v_x
\]
\[
0 = \dot{z}(t) = v_z
\]
\[
0 = \left( C_T - C_D \right) \frac{1}{2} \rho(z) V_{\text{cruise}}^2 S
\]
\[
0 = \frac{C_L}{2} \rho(z) V_{\text{cruise}}^2 S - mg(z)
\]

Introduction to Aeronautical Propulsion
Internal Combustion Reciprocating Engine

Linear motion of pistons converted to rotary motion to drive propeller

Turbojet Engines (1930s)

Thrust produced directly by exhaust gas

Axial-flow Turbojet (von Ohain, Germany)

Centrifugal-flow Turbojet (Whittle, UK)
Birth of the Jet Airplane

- **Heinkel He. 178 (1939)**
- **Gloster Meteor (1943)**
- **Messerschmitt Me 262 (1942)**
- **Gloster E/28/39 (1941)**

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- **Gloster Meteor (1943)**
- **Messerschmitt Me 262 (1942)**
- **Gloster E/28/39 (1941)**

Turbojet + Afterburner (1950s)

- Turbojet
  - Low-pressure compressor
  - High-pressure compressor
  - High-pressure turbine
  - Burner
  - Nozzle
  - Inlet

- Turbojet with Afterburner
  - Fuel spray bars
  - Flame holder
  - Afterburner duct
  - Adjustable nozzle
  - Dual rotation rates, N1 and N2, typical

Fuel added to exhaust

Additional air may be introduced
Turboprop Engines (1940s)

Exhaust gas drives propeller to produce thrust

High Bypass Ratio Turbofan

Aft-fan Engine

Geared Turbofan Engine

Propfan Engine
Ramjet and Scramjet

Ramjet (1940s)

Scramjet (1950s)

Electric Propulsion

Available Energy

Power Harvesting

Usable Power

Available Thrust

Energy Storage

Solar Cells

Fuel Cells

Turbine/Reciprocating Engines

Conversion Efficiency < 1

Power Conditioning

Transmission Inverters

Regulators

Generators

Gearing

Conditioning Efficiency < 1

Thrust Generation

Motors

Propellers

Ducted Fans

Propulsive Efficiency < 1
Electric Aircraft

- Solar cells/batteries
- Fuel cells
- Batteries

Hybrid-Electric Aircraft

- Turbine engines
- Generators
- Thrust
- Drag reduction
Thrust and Specific Impulse

\[ I_s = \frac{\text{Thrust}}{m g_e} \]  
Specific Impulse,  \( I_s \)  
Units = \( \frac{\text{m/s}}{\text{sec}} \)  
\( m \) = Mass flow rate of on-board propellant  
\( g_e \) = Gravitational acceleration at earth's surface

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Thrust and Thrust Coefficient

\[ \text{Thrust} \equiv C_T \frac{1}{2} \rho V^2 S \]

- Non-dimensional thrust coefficient, \( C_T \)
  - \( C_T \) is a function of power/throttle setting, fuel flow rate, blade angle, Mach number, ...
- Reference area, \( S \), may be
  - aircraft wing area,
  - propeller disk area, or
  - jet exhaust area
Sensitivity of Thrust to Airspeed

Nominal Thrust = $T_N \equiv C_{T_N} \frac{1}{2} \rho V_N^2 S$

$\left( \cdot \right)_N = \text{Nominal (or reference) value}$

Turbojet thrust is \textit{independent} of airspeed over wide range

Power

Assuming thrust is aligned with airspeed vector

$Power = P = \text{Thrust} \times \text{Velocity} \equiv C_T \frac{1}{2} \rho V^3 S$

Propeller–driven power is \textit{independent} of (subsonic) airspeed over a wide range
(reciprocating or turbine engine, with constant RPM or variable-pitch prop)
Next Time:
Low-Speed Aerodynamics

Reading:
Flight Dynamics
Aerodynamic Coefficients, 65–84

Supplementary Material
MATLAB Scripts for Flat-Earth Trajectory, No Aerodynamics

**Analytical Solution**

\[
g = 9.8; \\
t = 0:0.1:40; \\
vx0 = 10; \\
vz0 = 100; \\
x0 = 0; \\
z0 = 0; \\

vx1 = vx0; \\
vz1 = vz0 - gt; \\
x1 = x0 + vx0*t; \\
z1 = z0 + vz0*t - 0.5*gt.*t; \]

**Numerical Solution**

**Calling Routine**

\[
tspan = 40; \\
x0 = [10;100;0;0]; \\
[t1,x1] = ode45('FlatEarth',tspan,x0); \]

**Equations of Motion**

\[
function xdot = FlatEarth(t,x) \\
% x(1) = vx \\
% x(2) = vz \\
% x(3) = x \\
% x(4) = z \\
g = 9.8; \\
xdot(1) = 0; \\
xdot(2) = -g; \\
xdot(3) = x(1); \\
xdot(4) = x(2); \\
xdot = xdot';
end\]
Early Reciprocating Engines

• Rotary Engine:
  – Air-cooled
  – Crankshaft fixed
  – Cylinders turn with propeller
  – On/off control: No throttle

• V-8 Engine:
  – Water-cooled
  – Crankshaft turns with propeller

Reciprocating Engines

• Rotary
• In-Line
• Radial
• V-12
• Opposed
**Turbo-compound Reciprocating Engine**

- Exhaust gas drives the turbo-compressor
- Napier Nomad II shown (1949)

**Turbofan Engine (1960s)**

- Dual or triple rotation rates
Jet Engine Nacelles

Propeller-Driven Aircraft of the 1950s

**Reciprocating Engines**
- Douglas DC-6
- Douglas DC-7
- Lockheed Starliner (Constellation) 1649

**Turboprop Engines**
- Vickers Viscount
- Bristol Britannia
- Lockheed Electra
Pulsejet

Flapper-valved motor (1940s)

Dynajet Red Head (1950s)


Jet Transports of the 2000s

Airbus A380

Boeing 787

Embraer 195

Boeing 747-8F
SR-71: P&W J58
Variable-Cycle Engine (Late 1950s)

Hybrid Turbojet/Ramjet