Power and Thrust for Cruising Flight
Robert Stengel, Aircraft Flight Dynamics, MAE 331, 2018

U.S. Standard Atmosphere, 1976

Dynamic Pressure and Mach Number

\[ \rho = \text{air density}, \text{ function of height} \]
\[ = \rho_{\text{sealed}} e^{-\beta h} \]
\[ a = \text{speed of sound} \]
\[ = \text{linear function of height} \]

Dynamic pressure = \( \bar{q} \triangleq \rho V^2 / 2 \)
Mach number = \( V / a \)

Definitions of Airspeed

Airspeed is speed of aircraft measured with respect to air mass
Airspeed = Inertial speed if wind speed = 0

- **Indicated Airspeed (IAS)**

\[ IAS = \sqrt{2\left( p_{\text{stagnation}} - p_{\text{ambient}} \right)} / \rho_{\text{SL}} = \sqrt{2\left( p_{\text{total}} - p_{\text{static}} \right)} / \rho_{\text{SL}} \]
\[ \triangleq \frac{2q_c}{\rho_{\text{SL}}}, \text{ with } q_c \triangleq \text{impact pressure} \]

- **Calibrated Airspeed (CAS)**

\[ CAS = IAS \text{ corrected for instrument and position errors} \]
\[ = \sqrt{2\left( q_{c,\text{cor\#1}} \right) / \rho_{\text{SL}}} \]

Definitions of Airspeed

Airspeed is speed of aircraft measured with respect to air mass
Airspeed = Inertial speed if wind speed = 0

Equivalent Airspeed (EAS)*

\[ EAS = \text{CAS corrected for compressibility effects} = \sqrt{\frac{2(q_c)_{\text{corr}}^2}{\rho_{SL}}} \]

True Airspeed (TAS)*

\[ V \triangleq TAS = EAS \sqrt{\frac{\rho_{SL}}{\rho(z)}} = \text{IAS}_{\text{corrected}} \sqrt{\frac{\rho_{SL}}{\rho(z)}} \]

Mach number

\[ M = \frac{TAS}{a} \]

**Longitudinal Variables**

- Assume thrust is aligned with the velocity vector (small-angle approximation for $\alpha$)
- Mass = constant

\[
\begin{align*}
\dot{V} &= \frac{(C_T \cos \alpha - C_D) \frac{1}{2} \rho V^2 S - mg \sin \gamma}{m} = \frac{(C_T - C_D) \frac{1}{2} \rho V^2 S - mg \sin \gamma}{m} \\
\dot{V} &= \frac{(C_T \sin \alpha + C_L) \frac{1}{2} \rho V^2 S - mg \cos \gamma}{mV} = \frac{C_L \frac{1}{2} \rho V^2 S - mg \cos \gamma}{mV} \\
\dot{h} &= -\dot{z} = -v_z = V \sin \gamma \\
\dot{r} &= \dot{x} = v_x = V \cos \gamma
\end{align*}
\]

- $V = \text{velocity} = \text{Earth-relative airspeed}$
- $= \text{True airspeed with zero wind}$
- $\gamma = \text{flight path angle}$
- $h = \text{height (altitude)}$
- $r = \text{range}$
Conditions for Steady, Level Flight

- Flight path angle = 0
- Altitude = constant
- Airspeed = constant
- Dynamic pressure = constant

\[ 0 = \frac{\left( C_T - C_D \right) \frac{1}{2} \rho V^2 S}{m} \]

- Thrust = Drag

\[ 0 = \frac{\frac{1}{2} \rho V^2 S - m g}{m V} \]

- Lift = Weight

\[ \dot{h} = 0 \]

\[ \dot{r} = V \]

Power and Thrust

Propeller

\[ Power = P = T \times V = C_T \frac{1}{2} \rho V^3 S \approx \text{independent of airspeed} \]

Turbojet

\[ Thrust = T = C_T \frac{1}{2} \rho V^2 S \approx \text{independent of airspeed} \]

Throttle Effect

\[ T = T_{\text{max}} \delta T = \left[ C_{T_{\text{max}}} \bar{q} S \right] \delta T, \quad 0 \leq \delta T \leq 1 \]
Typical Effects of Altitude and Velocity on Power and Thrust

- Propeller [Air-breathing engine]
- Turbofan [In between]
- Turbojet
- Battery [Independent of altitude and airspeed]

Models for Altitude Effect on Turbofan Thrust

From *Flight Dynamics*, pp.117-118

\[
\text{Thrust} = C_T (V, \delta T) \frac{1}{2} \rho(h)V^2S
= \left[ (k_o + k_iV^n) \frac{1}{2} \rho(h)V^2S \right] \delta T, \text{ N}
\]

- \(k_o\) = Static thrust coefficient at sea level
- \(k_i\) = Velocity sensitivity of thrust coefficient
- \(n\) = Exponent of velocity sensitivity [\(n = -2\) for turbojet]
- \(\rho(h) = \rho_{sl} e^{-\beta h}\), \(\rho_{sl} = 1.225 \text{ kg/m}^3\), \(\beta = (1/9,042) \text{ m}^{-1}\)
Thrust of a Propeller-Driven Aircraft

With constant \( \text{rpm} \), variable-pitch propeller

\[
T = \eta_p \eta_I \frac{P_{\text{engine}}}{V} = \eta_{\text{net}} \frac{P_{\text{engine}}}{V}
\]

\( \eta_p = \text{propeller efficiency} \)
\( \eta_I = \text{ideal propulsive efficiency} \)
\( = \frac{TV}{T(V + \Delta V_{\text{inflow}})} = \frac{V}{V + \Delta V_{\text{inflow}}/2} \)
\( \eta_{\text{net}} = 0.85 - 0.9 \)

Efficiencies decrease with airspeed
Engine power decreases with altitude
Proportional to air density, w/o supercharger

Reciprocating-Engine Power and Specific Fuel Consumption (SFC)

\[
\frac{P(h)}{P_{\text{SL}}} = 1.132 \frac{\rho(h)}{\rho_{\text{SL}}} - 0.132
\]

\( SFC \propto \text{Independent of Altitude} \)

- Engine power decreases with altitude
  - Proportional to air density, w/o supercharger
  - Supercharger increases inlet manifold pressure, increasing power and extending maximum altitude

\( \text{Anderson (Torenbeek)} \)
**Propeller Efficiency, $\eta_P$, and Advance Ratio, $J$**

Advance Ratio

$$J = \frac{V}{nD}$$

where

$V = \text{airspeed, m/s}$

$n = \text{rotation rate, revolutions/s}$

$D = \text{propeller diameter, m}$

**Thrust of a Turbojet Engine**

$$T = \dot{m}V \left[ \left( \frac{\theta_o}{\theta_o - 1} \right) \left( \frac{\theta_i}{\theta_i - 1} \right) \left( \frac{\tau_c}{\theta_o \tau_c} - 1 \right) \right]^{1/2}$$

$$\dot{m} = \dot{m}_{\text{air}} + \dot{m}_{\text{fuel}}$$

$$\theta_o = \left( \frac{p_{\text{stag}}}{p_{\text{ambient}}} \right)^{(\gamma - 1)/\gamma} \quad \gamma = \text{ratio of specific heats} \approx 1.4$$

$$\theta_i = \left( \text{turbine inlet temp./freestream ambient temp.} \right)$$

$$\tau_c = \left( \text{compressor outlet temp./compressor inlet temp.} \right)$$

**Figure 8.18** Estimated propeller efficiency for the Piper Cherokee Arrow PA-28R.

from McCormick

Little change in thrust with airspeed below $M_{\text{crit}}$

Decrease with increasing altitude

from Kerrebrock
Electric Propulsion

Specific Energy and Energy Density of Fuel and Batteries (typical)

- Specific energy = energy/unit mass
- Energy density = energy/unit volume

<table>
<thead>
<tr>
<th>Energy Storage Material</th>
<th>Specific Energy, MJ/kg</th>
<th>Energy Density, MJ/L</th>
</tr>
</thead>
<tbody>
<tr>
<td>Lithium-Ion Battery</td>
<td>0.4-0.9</td>
<td>0.9-2.6</td>
</tr>
<tr>
<td>Jet Engine Fuel (Kerosene)</td>
<td>43</td>
<td>37</td>
</tr>
<tr>
<td>Gasoline</td>
<td>46</td>
<td>34</td>
</tr>
<tr>
<td>Methane (Liquified)</td>
<td>56</td>
<td>22</td>
</tr>
<tr>
<td>Hydrogen (Liquified)</td>
<td>142</td>
<td>9</td>
</tr>
</tbody>
</table>

Fuel cell energy conversion efficiency: 40-60%

Solar cell power conversion efficiency: 30-45%
Solar irradiance: 1 kW/m²
### Engine/Motor Power, Thrust, and Efficiency *(typical)*

<table>
<thead>
<tr>
<th>Engine/Motor Type</th>
<th>Power/Mass, kW/kg</th>
<th>Thermal Efficiency</th>
<th>Propulsive Efficiency</th>
</tr>
</thead>
<tbody>
<tr>
<td>Supercharged Radial Engine</td>
<td>1.8</td>
<td>25-50%</td>
<td>~Propeller Efficiency</td>
</tr>
<tr>
<td>Turboshaft Engine</td>
<td>5</td>
<td>40-60%</td>
<td>~Propeller Efficiency</td>
</tr>
<tr>
<td>Brushless DC Motor</td>
<td>1-2</td>
<td>-</td>
<td>~Propeller Efficiency</td>
</tr>
<tr>
<td>Turbojet Engine</td>
<td>10</td>
<td>40-60%</td>
<td>~1 – ( \frac{V_{exhaust}}{V \backslash V} )</td>
</tr>
<tr>
<td>Turbofan Engine</td>
<td>4-5</td>
<td>40-60%</td>
<td>~1 – ( \frac{V_{exhaust}}{V \backslash V} )</td>
</tr>
</tbody>
</table>

**Zunum ZA-10 Hybrid-Electric Aircraft**

- 12-passenger commuter aircraft (2023)
- Safran Ardiden 3Z turbine engine, 500kW (~ 650 shp)
- Lithium-ion batteries (TBD)
- Boeing and Jet Blue funding
- Goal: 610-nm (700-sm) range
- Turbo Commander test aircraft (2019)
Performance Parameters

Lift-to-Drag Ratio
\[ \frac{L}{D} = \frac{C_L}{C_D} \]

Load Factor
\[ n = \frac{L}{W} = \frac{L}{mg} \cdot \text{"g"} \cdot \text{s} \]

Thrust-to-Weight Ratio
\[ \frac{T}{W} = \frac{T}{mg} \cdot \text{"g"} \cdot \text{s} \]

Wing Loading
\[ \frac{W}{S}, \quad \text{N/m}^2 \text{ or lb/ft}^2 \]

Historical Factoid

• Aircraft Flight Distance Records

• Aircraft Flight Endurance Records
Steady, Level Flight

Trimmed Lift Coefficient, $C_L$

- Trimmed lift coefficient, $C_L$
  - Proportional to weight and wing loading factor
  - Decreases with $V^2$
  - At constant true airspeed, increases with altitude

$$W = C_{L_{trim}} \left( \frac{1}{2} \rho V^2 \right) S = C_{L_{trim}} \bar{q} S$$

$$C_{L_{trim}} = \frac{1}{\bar{q}} (W/S) = \frac{2}{\rho V^2} (W/S) = \left( \frac{2 e^{\beta h}}{\rho_0 V^2} \right) (W/S)$$

$\beta = 1/9,042$ m, inverse scale height of air density

$\rho_0$
Trimmed Angle of Attack, $\alpha$

- Trimmed angle of attack, $\alpha$
  - Constant if dynamic pressure and weight are constant
  - If dynamic pressure decreases, angle of attack must increase

\[
\alpha_{\text{trim}} = \frac{2W/\rho V^2 S - C_{L_{\alpha}}}{C_{L_{\alpha}}} = \frac{1}{q} \left( \frac{W}{S} - C_{L_{\alpha}} \right)
\]
Thrust Required for Steady, Level Flight

Trimmed thrust

\[ T_{\text{trim}} = D_{\text{cruise}} = C_{D_o} \left( \frac{1}{2} \rho V^2 S \right) + \varepsilon \frac{2W^2}{\rho V^2 S} \]

Minimum required thrust conditions

\[ \frac{\partial T_{\text{trim}}}{\partial V} = C_{D_o} (\rho V S) - \frac{4\varepsilon W^2}{\rho V^3 S} = 0 \]

Necessary Condition: Slope = 0

Necessary and Sufficient Conditions for Minimum Required Thrust

Necessary Condition = Zero Slope

\[ C_{D_o} (\rho V S) = \frac{4\varepsilon W^2}{\rho V^3 S} \]

Sufficient Condition for a Minimum = Positive Curvature when slope = 0

\[ \frac{\partial^2 T_{\text{trim}}}{\partial V^2} = C_{D_o} (\rho S) + \frac{12\varepsilon W^2}{\rho V^4 S} > 0 \]

(+)

(+)
Airspeed for Minimum Thrust in Steady, Level Flight

Satisfy necessary condition

\[ V^4 = \left( \frac{4\epsilon}{C_{D_o} \rho^2} \right) \left( \frac{W}{S} \right)^2 \]

Fourth-order equation for velocity

Choose the positive root

\[ V_{MT} = \sqrt[4]{\frac{2}{\rho} \left( \frac{W}{S} \right) \sqrt{\epsilon} \frac{1}{C_{D_o}}} \]

Lift, Drag, and Thrust Coefficients in Minimum-Thrust Cruising Flight

Lift coefficient

\[ C_{L_{MT}} = \frac{2}{\rho V_{MT}^2} \left( \frac{W}{S} \right) = \sqrt{\frac{C_{D_o}}{\epsilon}} = (C_L)_{(L/D)_{\text{max}}} \]

Drag and thrust coefficients

\[ C_{D_{MT}} = C_{D_o} + \epsilon C_{L_{MT}}^2 = C_{D_o} + \epsilon \left( \frac{C_{D_o}}{\epsilon} \right) = 2C_{D_o} = C_{T_{MT}} \]
Achievable Airspeeds in Constant-Altitude Flight

- Two equilibrium airspeeds for a given thrust or power setting
  - Low speed, high $C_L$, high $\alpha$
  - High speed, low $C_L$, low $\alpha$
- Achievable airspeeds between minimum and maximum values with maximum thrust or power

Power Required for Steady, Level Flight

\[ P = T \times V \]
Power Required for Steady, Level Flight

Trimmed power

\[ P_{\text{trim}} = T_{\text{trim}} \quad V = D_{\text{cruise}} \quad V = \left[ C_{D_0} \left( \frac{1}{2} \rho V^2 S \right) + \frac{2 \varepsilon W^2}{\rho V^2 S} \right] V \]

Minimum required power conditions

\[ \frac{\partial P_{\text{trim}}}{\partial V} = C_{D_0} \left( \frac{3}{2} \left( \rho V^2 S \right) - \frac{2 \varepsilon W^2}{\rho V^2 S} \right) = 0 \]

Airspeed for Minimum Power in Steady, Level Flight

- Satisfy necessary condition
  \[ C_{D_0} \frac{3}{2} \left( \rho V^2 S \right) = \frac{2 \varepsilon W^2}{\rho V^2 S} \]

- Fourth-order equation for velocity
  - Choose the positive root
  \[ V_{\text{MP}} = \sqrt{\frac{2 \left( \frac{W}{S} \right)}{\rho \left( \frac{1}{3} C_{D_0} \right)}} \]

- Corresponding lift and drag coefficients
  \[ C_{L_{\text{MP}}} = \sqrt{\frac{3 C_{D_0}}{\varepsilon}} \]
  \[ C_{D_{\text{MP}}} = 4 C_{D_0} \]
Achievable Airspeeds for Jet in Cruising Flight

Thrust = constant

\[ T_{\text{avail}} = C_D q S = C_{D_o} \left( \frac{1}{2} \rho V^2 S \right) + \frac{2 \varepsilon W^2}{\rho V^2 S} \]

\[ C_{D_o} \left( \frac{1}{2} \rho V^4 S \right) - T_{\text{avail}} V^2 + \frac{2 \varepsilon W^2}{\rho S} = 0 \]

\[ V^4 - \frac{2 T_{\text{avail}}}{C_{D_o} \rho S} V^2 + \frac{4 \varepsilon W^2}{C_{D_o} \left( \rho S \right)^2} = 0 \]

4th-order algebraic equation for \( V \)

Achievable Airspeeds for Jet in Cruising Flight

Solutions for \( V^2 \) can be put in quadratic form and solved easily

\[ V^2 \triangleq x; \quad V = \pm \sqrt{x} \]

\[ V^4 - \frac{2 T_{\text{avail}}}{C_{D_o} \rho S} V^2 + \frac{4 \varepsilon W^2}{C_{D_o} \left( \rho S \right)^2} = 0 \]

\[ x^2 + bx + c = 0 \]

\[ x = \frac{-b \pm \sqrt{\left( \frac{b}{2} \right)^2 - c}}{2} = V^2 \]
Thrust Required and Thrust Available for a Typical Bizjet

Available thrust decreases with altitude
Stall limitation at low speed
Mach number effect on lift and drag increases thrust required at high speed

**Typical Simplified Jet Thrust Model**

\[
T_{\text{max}}(h) = T_{\text{max}}(SL) \left( \frac{\rho(SL)e^{-\beta h}}{\rho(SL)} \right)^{\alpha} = T_{\text{max}}(SL)e^{-\beta h}
\]

Empirical correction to force thrust to zero at a given altitude, \(h_{\text{max}}\).
\(c\) is a convergence factor.

\[
T_{\text{max}}(h) = T_{\text{max}}(SL)e^{-x_{\beta}h} \left[ 1 - e^{-(h-h_{\text{max}})/c} \right]
\]

Thrust Required and Thrust Available for a Typical Bizjet
Next Time:
_Cruising Flight Envelope_

Supplemental Material
Models for Altitude Effect on Turbofan Thrust

From AeroModelMach.m in FLIGHT.m, Flight Dynamics, http://www.princeton.edu/~stengel/AeroModelMach.m

```matlab
[airDens,airPres,temp,soundSpeed] = Atmos(-x(6));
Thrust = u(4) * StaticThrust * (airDens / 1.225)^0.7 * (1 - exp((-x(6) – 17000)/2000));
```

Atmos(-x(6)): 1976 U.S. Standard Atmosphere function
-x(6) = h = Altitude, m
airDens = ρ = Air density at altitude h, kg/m³
u(4) = δT = Throttle setting, (0,1)

Empirical fit to match known characteristics of powerplant for generic business jet

```matlab
(airDens / 1.225)^0.7 * (1 - exp((-x(6) – 17000)/2000))
```

Hybrid-Electric Power System

Simulink Design Example

Hybrid Electric Aircraft Component Sizing
1. Plot battery status and currents (see code)
2. Battery (A/h): 100, 120, 140, 160, 180, 200
3. Aircraft: Single Seat, Double Seat
4. Flight Cycle: Height-Based, Time-Based
   Altitude: Low, High
5. Explore simulation results using scsptcourt
6. Sweep battery sizes (see code)
7. Sweep payload mass (see code)
8. Sweep battery and payload mass (see code)
9. Learn more about this example
Achievable Airspeeds in Propeller-Driven Cruising Flight

\[
V^4 - \frac{P_{\text{avail}}V}{C_D \rho S} + \frac{4 \varepsilon W^2}{C_{D_s} (\rho S)^2} = 0
\]

Solutions for \( V \) cannot be put in quadratic form; solution is more difficult, e.g., Ferrari’s method

\[
aV^4 + (0)V^3 + (0)V^2 + dV + e = 0
\]

Best bet: roots in MATLAB
**P-51 Mustang Minimum-Thrust Example**

Wing Span = 37 ft (9.83 m)
Wing Area = 235 ft² (21.83 m²)
Loaded Weight = 9,200 lb (3,465 kg)

\[ C_{D_0} = 0.0163 \]
\[ \varepsilon = 0.0576 \]
\[ \frac{W}{S} = 39.3 \text{ lb/ft² (1555.7 N/m²)} \]

**Airspeed for minimum thrust**

\[ V_{MT} = \sqrt{\frac{2}{\rho} \left( \frac{W}{S} \right) \left( \frac{\varepsilon}{C_{D_0}} \right) \left( \frac{2}{1555.7} \right)} = \sqrt{\frac{0.947}{0.0163}} \]
\[ = 76.49 \text{ m/s} \]

**P-51 Mustang Maximum L/D Example**

Wing Span = 37 ft (9.83 m)
Wing Area = 235 ft² (21.83 m²)
Loaded Weight = 9,200 lb (3,465 kg)

\[ C_{D_0} = 0.0163 \]
\[ \varepsilon = 0.0576 \]
\[ \frac{W}{S} = 1555.7 \text{ N/m²} \]

\[ C_{L_{\text{max}}} = \frac{C_{D_0}}{\varepsilon} = C_{\text{MT}} = 0.0326 \]

\[ (L/D)_{\text{max}} = \frac{1}{2 \sqrt{\varepsilon C_{D_0}}} = 16.31 \]

\[ V_{L/D_{\text{max}}} = \frac{V_{\text{MT}}}{\sqrt{\rho}} = \frac{76.49}{\sqrt{1555.7}} \text{ m/s} \]