Cruising Flight Envelope
Robert Stengel, Aircraft Flight Dynamics, MAE 331, 2018

Learning Objectives
- Definitions of airspeed
- Performance parameters
- Steady cruising flight conditions
- Breguet range equations
- Optimize cruising flight for minimum thrust and power
- Flight envelope

The Flight Envelope
Flight Envelope Determined by Available Thrust

- All altitudes and airspeeds at which an aircraft can fly
  - in steady, level flight
  - at fixed weight

  - Flight ceiling defined by available climb rate
    - Absolute: 0 ft/min
    - Service: 100 ft/min
    - Performance: 200 ft/min

Excess thrust provides the ability to accelerate or climb

Additional Factors Define the Flight Envelope

- Maximum Mach number
- Maximum allowable aerodynamic heating
- Maximum thrust
- Maximum dynamic pressure
- Performance ceiling
- Wing stall
- Flow-separation buffet
  - Angle of attack
  - Local shock waves
Boeing 787 Flight Envelope
(HW #5, 2008)

Flight Envelope of B-787-8

- Service/Performance Ceiling
- Stall Cruise Region
- Speed of Sound
- Dynamic Pressure = 15,000 N/m²

Lockheed U-2 “Coffin Corner”

Stall buffeting and Mach buffeting are limiting factors
Narrow corridor for safe flight
**Historical Factoids**

**Air Commerce Act of 1926**

- Airlines formed to carry mail and passengers:
  - Northwest (1926)
  - Eastern (1927), bankruptcy
  - Pan Am (1927), bankruptcy
  - Boeing Air Transport (1927), became United (1931)
  - Delta (1928), consolidated with Northwest, 2010
  - American (1930)
  - TWA (1930), acquired by American
  - Continental (1934), consolidated with United, 2010

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**Commercial Aircraft of the 1930s**

- **Streamlining, engine cowlings**
  - Douglas DC-1, DC-2, DC-3
  - Lockheed 14 Super Electra, Boeing 247

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[Video link: http://www.youtube.com/watch?v=3a8G87qnZz4]
Comfort and Elegance by the End of the Decade

*Boeing 307, 1st pressurized cabin (1936), flight engineer, B-17 precursor, large dorsal fin (exterior and interior)*

Sleeping bunks on transcontinental planes (e.g., *DC-3*)
Full-size dining rooms on flying boats

*Optimal Cruising Flight*
Maximum Lift-to-Drag Ratio

Lift-to-drag ratio

\[ \frac{L}{D} = \frac{C_L}{C_D} = \frac{C_L}{C_{D_0} + \epsilon C_L^2} \]

Satisfy necessary condition for a maximum

\[ \frac{\partial}{\partial C_L} \left( \frac{C_L}{C_{D_0}} \right) = \frac{1}{C_{D_0} + \epsilon C_L^2} - \frac{2\epsilon C_L^2}{(C_{D_0} + \epsilon C_L^2)^2} = 0 \]

Lift coefficient for maximum \( \frac{L}{D} \) and minimum thrust are the same

\[ \left( C_L \right)_{\text{L/D max}} = \frac{C_{D_0}}{\epsilon} = C_{L_{MT}} \]

Airspeed, Drag Coefficient, and Lift-to-Drag Ratio for \( \frac{L}{D}\text{max} \)

Airspeed

\[ V_{\text{L/D max}} = V_{MT} = \sqrt{\frac{2}{\rho} \left( \frac{W}{S} \right) \frac{\epsilon}{C_{D_0}}} \]

Drag Coefficient

\[ \left( C_D \right)_{\text{L/D max}} = C_{D_0} + 2C_{D_0} = 3C_{D_0} \]

Maximum L/D

\[ \left( \frac{L}{D} \right)_{\text{max}} = \frac{\sqrt{C_{D_0}/\epsilon}}{2C_{D_0}} = \frac{1}{2\sqrt{\epsilon C_{D_0}}} \]

Maximum \( \frac{L}{D} \) depends only on induced drag factor and zero-lift drag coefficient.
Induced drag factor and zero-lift drag coefficient are functions of Mach number.
Cruising Range and Specific Fuel Consumption

- **Thrust = Drag**
  
  \[ 0 = (C_T - C_D) \frac{1}{2} \rho V^2 S / m \]

- **Lift = Weight**
  
  \[ 0 = \left( C_l \frac{1}{2} \rho V^2 s - m g \right) / m V \]

- **Thrust specific fuel consumption,** \( TSFC = c_T \)
  
  Fuel mass burned per sec per unit of thrust

  \[ c_T : \frac{kg/s}{kN} \quad \dot{m}_f = -c_T T \]

- **Power specific fuel consumption,** \( PSFC = c_P \)
  
  Fuel mass burned per sec per unit of power

  \[ c_P : \frac{kg/s}{kW} \quad \dot{m}_f = -c_P P \]

Level flight

\[ h = 0 \]

\[ \dot{r} = V \]

Breguet Range Equation for Jet Aircraft

Rate of change of range with respect to weight of fuel burned

\[
\frac{dr}{dm} = \frac{dr}{dt} \frac{dt}{dm} = \dot{r} = \frac{V}{c_T} = -\frac{V}{c_T D} = -\left( \frac{L}{D} \right) \frac{V}{c_T mg}
\]

\[
dr = -\left( \frac{L}{D} \right) \frac{V}{c_T mg} dm
\]

**Range traveled**

\[
Range = R = \int_0^r dr = -\int_{W_i}^{W_f} \left( \frac{L}{D} \right) \left( \frac{V}{c_T g} \right) \frac{dm}{m}
\]
Maximum Range of a Jet Aircraft Flying at Constant Altitude

At constant altitude and SFC

\[ V_{\text{cruise}}(t) = \sqrt{2W(t)/C_L \rho(h_{\text{fuel}})} S \]

\[ \text{Range} = -\int_{W_i}^{W_f} \left( \frac{C_L}{C_D} \left( \frac{1}{c_T g} \right) \sqrt{2 \frac{2}{C_L \rho S}} \frac{dm}{m^{1/2}} \right) \]

\[ = \left( \frac{\sqrt{C_L}}{C_D} \right) \left( \frac{2}{c_T g} \right) \sqrt{2 \rho S} (m_i^{1/2} - m_f^{1/2}) \]

Range is maximized when

\[ \left( \frac{\sqrt{C_L}}{C_D} \right) = \text{maximum} \]

Breguet Range Equation for Jet Aircraft at Constant Airspeed

For constant true airspeed, \( V = V_{\text{cruise}} \), and SFC

\[ R = -\left( \frac{L}{D} \right) \left( \frac{V_{\text{cruise}}}{c_T g} \right) \ln\left(m_i\right)_{m_i} \]

\[ = \left( \frac{L}{D} \right) \left( \frac{V_{\text{cruise}}}{c_T g} \right) \ln\left(\frac{m_i}{m_f}\right) \]

- \( V_{\text{cruise}}(C_L/C_D) \) as large as possible
- \( M \rightarrow M_{\text{crit}} \)
- \( \rho \) as small as possible
- \( h \) as high as possible
Maximize Jet Aircraft Range Using Optimal Cruise-Climb

\[ \frac{\partial R}{\partial C_L} \propto \frac{\partial}{\partial C_L} \left( \frac{V_{\text{cruise}}}{C_L} \right) = \frac{\partial}{\partial C_L} \left( \frac{V_{\text{cruise}}}{(C_{D_o} + \epsilon C_L^2)} \right) = 0 \]

\[ V_{\text{cruise}} = \sqrt{2W/C_L \rho S} \]

Assume \( \sqrt{2W(t)/\rho(h)S} \) = constant

\( i.e., \) airplane climbs at constant TAS as fuel is burned

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Maximize Jet Aircraft Range Using Optimal Cruise-Climb

\[ \frac{\partial}{\partial C_L} \left[ \frac{V_{\text{cruise}} C_L}{(C_{D_o} + \epsilon C_L^2)} \right] = \frac{2W}{\rho S} \frac{\partial}{\partial C_L} \left[ \frac{C_L^{1/2}}{(C_{D_o} + \epsilon C_L^2)} \right] = 0 \]

**Optimal values**: (see Supplemental Material)

\[ C_{L_{\text{max}}} = \sqrt{\frac{C_{D_o}}{3\epsilon}} : \text{Lift Coefficient for Maximum Range} \]

\[ C_{D_{\text{max}}} = C_{D_o} + \frac{C_{D_o}}{3} = \frac{4}{3} C_{D_o} \]

\[ V_{\text{cruise-climb}} = \sqrt{2W(t)/C_{L_{\text{max}}} \rho(h)S} = a(h) M_{\text{cruise-climb}} \]

\( a(h) \): Speed of sound; \( M_{\text{cruise-climb}} \): Mach number
Step-Climb Approximates Optimal Cruise-Climb

- **Cruise-climb** usually violates air traffic control rules
- Constant-altitude cruise does not
- Compromise: **Step climb** from one allowed altitude to the next as fuel is burned

**Historical Factoid**

- Louis Breguet (1880-1955), aviation pioneer
  - Gyroplane (1905), flew vertically in 1907
  - Breguet Type 1 (1909), fixed-wing aircraft
  - Formed Compagnie des messageries aériennes (1919), predecessor of Air France
- Breguet Aviation: built numerous aircraft until after World War II; teamed with BAC in SEPECAT (1966)
  - Merged with Dassault in 1971
Next Time:
Gliding, Climbing, and
Turning Flight

Supplemental Material
Seaplanes Became the First TransOceanic Air Transports

- PanAm led the way
  - 1st scheduled TransPacific flights (1935)
  - 1st scheduled TransAtlantic flights (1938)
  - 1st scheduled non-stop Trans-Atlantic flights (VS-44, 1939)
- Boeing B-314, Vought-Sikorsky VS-44, Shorts Solent
- Superseded by more efficient landplanes (lighter, less drag)

Maximize Jet Aircraft Range Using Optimal Cruise-Climb

\[
\frac{\partial}{\partial \alpha} \frac{V_{\text{cruise}} C_L}{(C_{D_o} + C_{L}^2 \varepsilon)} = \frac{2w}{\sqrt{\rho S}} \frac{\partial}{\partial \alpha} \frac{C_{L}^{1/2}}{(C_{D_o} + C_{L}^2 \varepsilon)} = 0
\]

\[
\frac{2w}{\sqrt{\rho S}} = \text{Constant}; \quad \text{let } C_L^{1/2} = x, \quad C_L = x^2
\]

\[
\frac{\partial}{\partial x} \left[ \frac{x}{(C_{D_o} + \varepsilon x^4)} \right] = \frac{(C_{D_o} + \varepsilon x^4) - x(4\varepsilon x^3)}{(C_{D_o} + \varepsilon x^4)^2} = \frac{C_{D_o} - 3\varepsilon x^4}{(C_{D_o} + \varepsilon x^4)^2}
\]

Optimal values:
\[
C_{L_{\text{opt}}} = \sqrt{\frac{C_{D_o}}{3\varepsilon}}; \quad C_{D_{\text{opt}}} = C_{D_o} + \frac{C_{D_o}}{3} = \frac{4}{3} C_{D_o}
\]
Breguet Range Equation for Propeller-Driven Aircraft

Rate of change of range with respect to weight of fuel burned

\[
\frac{dr}{dw} = \frac{\dot{r}}{\dot{w}} = \frac{V}{(-c_pP)} = \frac{V}{c_pTV} = \frac{V}{c_pDV} = \left(\frac{L}{D}\right) \frac{1}{c_pW}
\]

Range traveled

\[
Range = R = \int_0^R dr = \int_{W_i}^{W_f} \left(\frac{L}{D}\right) \left(\frac{1}{c_p}\right) \frac{dw}{w}
\]

Breguet Range Equation for Propeller-Driven Aircraft

For constant true airspeed, \( V = V_{cruise} \)

\[
R = -\left(\frac{L}{D}\right) \left(\frac{1}{c_p}\right) \ln\left(\frac{W_i}{W_f}\right) = \left(\frac{C_L}{C_D}\right) \left(\frac{1}{c_p}\right) \ln\left(\frac{W_i}{W_f}\right)
\]

Range is maximized when

\[
\left(\frac{C_L}{C_D}\right) = \text{maximum} = \left(\frac{L}{D}\right)_{\text{max}}
\]
Achievable Airspeeds in Propeller-Driven Cruising Flight

Power = constant

\[ P_{\text{avail}} = T_{\text{avail}} V \]

\[ V^4 - \frac{P_{\text{avail}} V}{C_D \rho S} + \frac{4 \varepsilon W^2}{C_D \left( \rho S \right)^2} = 0 \]

Solutions for \( V \) cannot be put in quadratic form; solution is more difficult, e.g., Ferrari’s method

\[ aV^4 + (0)V^3 + (0)V^2 + dV + e = 0 \]

Best bet: roots in MATLAB

P-51 Mustang Minimum-Thrust Example

Wing Span = 37 ft (9.83 m)
Wing Area = 235 ft\(^2\) (21.83 m\(^2\))
Loaded Weight = 9,200 lb (3,465 kg)
\[ C_D = 0.0163 \]
\[ \varepsilon = 0.0576 \]
\[ W/S = 39.3 \text{ lb} / \text{ft}^2 \ (1555.7 \text{ N} / \text{m}^2) \]

\[ V_{\text{MR}} = \sqrt{\frac{2W}{\rho S} / \sqrt{C_D \varepsilon}} = \sqrt{\frac{2 \times 1555.7}{0.0163}} = 76.49 \text{ m/s} \]

Altitude, m  Air Density, kg/m\(^3\)  \( \rho_{\text{atm}} \), m/s
0 1.23 69.11
2,500 0.96 78.30
5,000 0.74 89.15
10,000 0.41 118.87
P-51 Mustang
Maximum L/D Example

\[
(C_D)_{L/D_{\text{max}}} = 2C_{D_0} = 0.0326
\]

\[
(C_L)_{L/D_{\text{max}}} = \sqrt{\frac{C_{D_{\text{max}}}}{\varepsilon}} = C_{l_{\text{opt}}} = 0.531
\]

\[
(L / D)_{\text{max}} = \frac{1}{2\sqrt{\varepsilon C_{D_0}}} = 16.31
\]

\[
V_{L/D_{\text{max}}} = V_{\text{MT}} = \frac{76.49}{\sqrt{\rho}} \text{ m/s}
\]

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<tr>
<th>Altitude, m</th>
<th>Air Density, kg/m^3</th>
<th>V_{\text{MT}}, m/s</th>
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Wing Span = 37 ft (9.83 m)
Wing Area = 235 ft² (21.83 m²)
Loaded Weight = 9,200 lb (3,465 kg)
\[C_{D_0} = 0.0163\]
\[\varepsilon = 0.0576\]
\[W / S = 1555.7 \text{ N/} m^2\]

P-51 Mustang
Maximum Range
(Internal Tanks only)

\[
W = C_{l_{\text{max}}} \bar{q} S
\]

\[
C_{l_{\text{max}}} = \frac{1}{\bar{q}} (W / S)
\]

\[
= \frac{2}{\rho V^2} (W / S) = \left( \frac{2 \rho V^2}{\rho V T} \right) (W / S)
\]

\[
R = \left( \frac{C_L}{C_D}_{\text{max}} \right) \left( \frac{1}{C_F} \right) \ln \left( \frac{W}{W_F} \right)
\]

\[
= (16.31) \left( \frac{1}{0.0017} \right) \ln \left( \frac{3,465 + 600}{3,465} \right)
\]

\[
= 1,530 \text{ km (825 nm)}
\]