Gliding, Climbing, and Turning Flight Performance
Robert Stengel, Aircraft Flight Dynamics, MAE 331, 2018

Learning Objectives

• Conditions for gliding flight
• Parameters for maximizing climb angle and rate
• Review the V-n diagram
• Energy height and specific excess power
• Alternative expressions for steady turning flight
• The Herbst maneuver

Reading:
Flight Dynamics
Aerodynamic Coefficients, 130-141

Review Questions

• How does air density decrease with altitude?
• What are the different definitions of airspeed?
• What is a “lift-drag polar”?
• Power and thrust: How do they vary with altitude?
• What factors define the “flight envelope”?
• What were some features of the first commercial transport aircraft?
• What are the important parameters of the “Breguet Range Equation”?
• What is a “step climb”, and why is it important?
Gliding Flight

Equilibrium Gliding Flight

\[ C_D \frac{1}{2} \rho V^2 S = -W \sin \gamma \]
\[ C_L \frac{1}{2} \rho V^2 S = W \cos \gamma \]
\[ \dot{h} = V \sin \gamma \]
\[ \dot{r} = V \cos \gamma \]
Gliding Flight

- Thrust = 0
- Flight path angle < 0 in gliding flight
- Altitude is decreasing
- Airspeed ~ constant
- Air density ~ constant

### Gliding flight path angle

\[
\tan \gamma = -\frac{D}{L} = -\frac{C_D}{C_L} = \frac{\dot{h}}{\dot{r}} = \frac{dh}{dr}; \quad \gamma = -\tan^{-1} \left( \frac{D}{L} \right) = -\cot^{-1} \left( \frac{L}{D} \right)
\]

### Corresponding airspeed

\[
V_{glide} = \sqrt{\frac{2W}{\rho S \sqrt{C_D^2 + C_L^2}}}
\]

Maximum Steady Gliding Range

- Glide range is maximum when \( \gamma \) is least negative, i.e., most positive
- This occurs at \( (L/D)_{max} \)
Maximum Steady Gliding Range

- Glide range is maximum when $\gamma$ is least negative, i.e., most positive
- This occurs at $(L/D)_{\text{max}}$

$$\gamma_{\text{max}} = -\tan^{-1}\left(\frac{D}{L}\right)_{\text{min}} = -\cot^{-1}\left(\frac{L}{D}\right)_{\text{max}}$$

$$\tan \gamma = \frac{\dot{h}}{\dot{r}} = \text{negative constant} = \frac{(h - h_o)}{(r - r_o)}$$

$$\Delta r = \frac{\Delta h}{\tan \gamma} = -\Delta h = \text{maximum when } \frac{L}{D} = \text{maximum}$$

Sink Rate, m/s

Lift and drag define $\gamma$ and $V$ in gliding equilibrium

$$D = C_D \frac{1}{2} \rho V^2 S = -W \sin \gamma$$

$$\sin \gamma = -\frac{D}{W}$$

$$L = C_L \frac{1}{2} \rho V^2 S = W \cos \gamma$$

$$V = \sqrt{\frac{2W \cos \gamma}{C_L \rho S}}$$

Sink rate = altitude rate, $\frac{dh}{dt}$ (negative)

$$\dot{h} = V \sin \gamma$$

$$= -\sqrt{\frac{2W \cos \gamma}{C_L \rho S}} \left(\frac{D}{W}\right) = -\sqrt{\frac{2W \cos \gamma}{C_L \rho S}} \left(\frac{L}{W}\right) \left(\frac{D}{L}\right)$$

$$= -\sqrt{\frac{2W \cos \gamma}{C_L \rho S}} \cos \gamma \left(\frac{1}{L/D}\right)$$
Conditions for Minimum Steady Sink Rate

- Minimum sink rate provides maximum endurance
- Minimize sink rate by setting \( \partial(h/dt)/\partial C_L = 0 \) (\( \cos \gamma \sim 1 \))

\[
\dot{h} = -\sqrt{\frac{2W \cos \gamma}{C_L \rho S} \cos \gamma \left( \frac{C_D}{C_L} \right)}
\]

\[
\approx -\sqrt{\frac{2W \cos^3 \gamma}{\rho S} \left( \frac{C_D}{C_L^{3/2}} \right)} \approx -\sqrt{\frac{2W}{S} \left( \frac{C_D}{C_L^{3/2}} \right)}
\]

\[
C_{L_{ME}} = \sqrt{\frac{3C_D}{\epsilon}} \quad \text{and} \quad C_{D_{ME}} = 4C_D
\]

\[L/D\] and \( V_{ME} \) for Minimum Sink Rate

\[
(L/D)_{ME} = \frac{1}{4} \sqrt{\frac{3}{\epsilon C_D}} = \frac{\sqrt{3}}{2} (L/D)_{max} \approx 0.86 (L/D)_{max}
\]

\[
V_{ME} = \sqrt{\frac{2W}{\rho S \sqrt{C_{D_{ME}}^2 + C_{L_{ME}}^2}}} \approx \sqrt{\frac{2(W/S)}{\rho} \sqrt{\frac{\epsilon}{3C_D}}} \approx 0.76 V_{L/D_{max}}
\]
L/D for Minimum Sink Rate

• For $L/D < L/D_{\text{max}}$, there are two solutions
• Which one produces smaller sink rate?

\[
\frac{L/D}{ME} \approx 0.86 \left(\frac{L/D}{max}\right)
\]

\[
V_{ME} \approx 0.76 V_{L/D_{\text{max}}}
\]

Historical Factoids
Lifting-Body Reentry Vehicles

M2-F1  
M2-F2  
M2-F3  
HL-10  
X-24A  
X-24B  
M2-F2

Milestones in Flight History
Dryden Flight Research Center
NASA

Experience Lateral Oscillations in Flight
Circa 1967
Climbing Flight

- Flight path angle
  \[ V = 0 = \frac{(T - D - W \sin \gamma)}{m} \]
  \[ \sin \gamma = \frac{(T - D)}{W} ; \quad \gamma = \sin^{-1} \left( \frac{T - D}{W} \right) \]

- Required lift
  \[ \dot{V} = 0 = \frac{(L - W \cos \gamma)}{mV} \]
  \[ L = W \cos \gamma \]

Rate of climb, \( dh/dt \) = Specific Excess Power

\[ \dot{h} = V \sin \gamma = V \frac{(T - D)}{W} = \left( \frac{P_{\text{thrust}} - P_{\text{drag}}}{W} \right) \]

**Specific Excess Power (SEP)** = \( \frac{\text{Excess Power}}{\text{Unit Weight}} = \frac{P_{\text{thrust}} - P_{\text{drag}}}{W} \)
Steady Rate of Climb

Climb rate

\[ \dot{h} = V \sin \gamma = V \left[ \left( \frac{T}{W} \right) - \left( \frac{C_D + \varepsilon C_L^2}{(W/S)} \right) \right] \]

Note significance of thrust-to-weight ratio and wing loading

\[ \dot{h} = V \left[ \left( \frac{T}{W} \right) - \frac{C_D}{(W/S)} - \frac{\varepsilon(W/S)\cos^2 \gamma}{\bar{q}} \right] \]

\[ = V \left( \frac{T(h)}{W} \right) - \frac{C_D \rho(h)V^3}{2(W/S)} - \frac{2\varepsilon(W/S)\cos^2 \gamma}{\rho(h)V} \]

Condition for Maximum Steady Rate of Climb

Necessary condition for a maximum with respect to airspeed

\[ \frac{\partial \dot{h}}{\partial V} = 0 = \left[ \left( \frac{T}{W} \right) + V \left( \frac{\partial T / \partial V}{W} \right) \right] - \frac{3C_D \rho V^2}{2(W/S)} + \frac{2\varepsilon(W/S)\cos^2 \gamma}{\rho V^2} \]
Maximum Steady Rate of Climb: Propeller-Driven Aircraft

True Airspeed

- At constant power
  \[ \frac{\partial P_{\text{thrust}}}{\partial V} = 0 = \left[ \frac{T}{W} + V \left( \frac{\partial T}{\partial V} / W \right) \right] \]

- With \( \cos^2 \gamma \sim 1 \), optimality condition reduces to
  \[ \frac{\partial h}{\partial V} = 0 = -\frac{3C_D \rho V^2}{2(W/S)} + \frac{2 \varepsilon (W/S)}{\rho V^2} \]

- Airspeed for maximum rate of climb at maximum power, \( P_{\text{max}} \)
  \[ V^4 = \left( \frac{4}{3} \right) \varepsilon \left( \frac{W}{S} \right)^2 + \frac{2 \varepsilon (W/S)}{\rho} \sqrt{\frac{\varepsilon}{3C_D}} = V_{ME} \]

Maximum Steady Rate of Climb: Jet-Driven Aircraft

True Airspeed

Condition for a maximum at constant thrust and \( \cos^2 \gamma \sim 1 \)

\[ \frac{\partial h}{\partial V} = 0 \]

\[ -\frac{3C_D \rho}{2(W/S)} V^4 + \left( \frac{T}{W} \right) V^2 + \frac{2 \varepsilon (W/S)}{\rho} = 0 \]

\[ -\frac{3C_D \rho}{2(W/S)} (V^2)^2 + \left( \frac{T}{W} \right) (V^2) + \frac{2 \varepsilon (W/S)}{\rho} = 0 \]

Quadratic in \( V^2 \)

Airspeed for maximum rate of climb at maximum thrust, \( T_{\text{max}} \)

\[ 0 = ax^2 + bx + c \quad \text{and} \quad V = +\sqrt{x} \]
Optimal Climbing Flight

What is the Fastest Way to Climb from One Flight Condition to Another?
Energy Height

- Specific Energy
  - = (Potential + Kinetic Energy) per Unit Weight
  - = Energy Height

$\text{Specific Energy} \equiv \frac{\text{Total Energy}}{\text{Unit Weight}}$

$= \frac{mgh + \frac{mV^2}{2}}{mg} = h + \frac{V^2}{2g}$

$\equiv \text{Energy Height, } E_h, \text{ft or m}$

Can trade altitude for airspeed with no change in energy height if thrust and drag are zero

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Specific Excess Power

Rate of change of Specific Energy

$\frac{dE_h}{dt} = \frac{d}{dt} \left( h + \frac{V^2}{2g} \right) = \frac{dh}{dt} + \left( \frac{V}{g} \right) \frac{dV}{dt}$

$= V \sin \gamma + \left( \frac{V}{g} \right) \left( \frac{T - D - mg \sin \gamma}{m} \right) = V \frac{(T - D)}{W}$

$= \text{Specific Excess Power (SEP)}$

$\text{Excess Power} \equiv \frac{P_{\text{thrust}} - P_{\text{drag}}}{\text{Unit Weight}} = \frac{(C_T - C_D) \frac{1}{2} \rho(h)V^2S}{W}$
Contours of Constant Specific Excess Power

- Specific Excess Power is a function of altitude and airspeed
- $SEP$ is maximized at each altitude, $h$, when

$$\max_{h \in V} SEP(h) \quad \frac{d}{dV} SEP(h) = 0$$

Subsonic Minimum-Time Energy Climb

**Objective:** Minimize time to climb to desired altitude and airspeed

**Minimum-Time Strategy:**
- Zoom climb/diving to intercept $SEP_{\text{max}}(h)$ contour
- Climb at $SEP_{\text{max}}(h)$
- Zoom climb/diving to intercept target $SEP_{\text{max}}(h)$ contour

*Bryson, Desai, Hoffman, 1969*
**Subsonic Minimum-Fuel Energy Climb**

**Objective:** Minimize fuel to climb to desired altitude and airspeed

- **Minimum-Fuel Strategy:**
  - Zoom climb/dive to intercept \([SEP(h)/(dm/dt)]_{\text{max}}\) contour
  - Climb at \([SEP(h)/(dm/dt)]_{\text{max}}\)
  - Zoom climb/dive to intercept target \([SEP(h)/(dm/dt)]_{\text{max}}\) contour

*Bryson, Desai, Hoffman, 1969*

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**Supersonic Minimum-Time Energy Climb**

**Objective:** Minimize time to climb to desired altitude and airspeed

- **Minimum-Time Strategy:**
  - Intercept subsonic \(SEP_{\text{max}}(h)\) contour
  - Climb at \(SEP_{\text{max}}(h)\) to intercept matching zoom climb/dive contour
  - Zoom climb/dive to intercept supersonic \(SEP_{\text{max}}(h)\) contour
  - Climb at \(SEP_{\text{max}}(h)\) to intercept target \(SEP_{\text{max}}(h)\) contour
  - Zoom climb/dive to intercept target \(SEP_{\text{max}}(h)\) contour

*Bryson, Desai, Hoffman, 1969*
Checklist

- Energy height?
- Contours?
- Subsonic minimum-time climb?
- Supersonic minimum-time climb?
- Minimum-fuel climb?

\[
\frac{dE_h}{dm_{fuel}} = \frac{dE_h}{dt} \frac{dt}{dm_{fuel}} = \frac{1}{m_{fuel}} \left[ \frac{dh}{dt} + \left( \frac{V}{g} \right) \frac{dV}{dt} \right]
\]
SpaceShipOne Altitude vs. Range
MAE 331 Assignment #4, 2010

SpaceShipOne State Histories
SpaceShipOne Dynamic Pressure and Mach Number Histories

The Maneuvering Envelope
Maneuvering Envelope:

- **Maneuvering envelope**: limits on normal load factor and allowable equivalent airspeed
  - Structural factors
  - Maximum and minimum achievable lift coefficients
  - Maximum and minimum airspeeds
  - Protection against over stressing due to gusts
  - Corner Velocity: Intersection of maximum lift coefficient and maximum load factor

- **Typical positive load factor limits**
  - Transport: > 2.5
  - Utility: > 4.4
  - Aerobatic: > 6.3
  - Fighter: > 9

- **Typical negative load factor limits**
  - Transport: < -1
  - Others: < -1 to -3

Maneuvering Envelopes (*V-n Diagrams*)
for Three Fighters of the Korean War Era

- Republic F-84
- North American F-86
- Lockheed F-94
**Turning Flight**

- Level flight = constant altitude
- Sideslip angle = 0
- Vertical force equilibrium

\[ L \cos \mu = W \]

- Load factor

\[ n = \frac{L}{W} = \frac{L}{mg} = \sec \mu, \text{"} g \text{"} \]

- Thrust required to maintain level flight

\[ T_{req} = (C_{D_0} + \varepsilon C_L^2) \frac{1}{2} \rho V^2 S = D_0 + \frac{2\varepsilon}{\rho V^2 S} \left( \frac{W}{\cos \mu} \right)^2 \]

\[ = D_0 + \frac{2\varepsilon}{\rho V^2 S} (nW)^2 \]
Maximum Bank Angle in Steady Level Flight

**Bank angle**

\[
\cos \mu = \frac{W}{C_L q S}
\]
\[
= \frac{1}{n}
\]
\[
= W \left( \frac{2\epsilon}{T_{req} - D_v} \rho V^2 S \right)
\]

Bank angle is limited by

\[
C_{I_{max}} \text{ or } T_{max} \text{ or } n_{max}
\]

Turning Rate and Radius in Level Flight

**Turning rate**

\[
\hat{\xi} = \frac{C_L q S \sin \mu}{mV}
\]
\[
= \frac{W \tan \mu}{mV}
\]
\[
= \frac{g \tan \mu}{V}
\]
\[
= \frac{\sqrt{L^2 - W^2}}{mV}
\]
\[
= \frac{W \sqrt{n^2 - 1}}{mV}
\]
\[
= \sqrt{\frac{T_{req} - D_v}{\rho V^2 S/2\epsilon - W^2}}
\]

Turning rate is limited by

\[
C_{I_{max}} \text{ or } T_{max} \text{ or } n_{max}
\]

**Turning radius**

\[
R_{turn} = \frac{V}{\hat{\xi}} = \frac{V^2}{g \sqrt{n^2 - 1}}
\]
Maximum Turn Rates

- Corner velocity
- Normal acceleration limit
- Maximum sustainable turn rate
- “Wind-up turns”
- Lift coefficient limit
- Thrust limit

Corner Velocity Turn

- Corner velocity
  \[ V_{\text{corner}} = \frac{2n_{\text{max}}W}{\sqrt{C_{\text{L}_{\text{max}}} \rho S}} \]
- For steady climbing or diving flight
  \[ \sin \gamma = \frac{T_{\text{max}} - D}{W} \]
- Turning radius
  \[ R_{\text{turn}} = \frac{V^2 \cos^2 \gamma}{g \sqrt{n_{\text{max}}^2 \cos^2 \gamma}} \]
Corner Velocity Turn

- **Turning rate**

\[ \dot{\gamma} = \sqrt{\frac{g \left(n_{\text{max}}^2 - \cos^2 \gamma\right)}{V \cos \gamma}} \]

- **Time to complete a full circle**

\[ t_{2\pi} = \frac{V \cos \gamma}{g \sqrt{n_{\text{max}}^2 - \cos^2 \gamma}} \]

- **Altitude gain/loss**

\[ \Delta h_{2\pi} = t_{2\pi} V \sin \gamma \]

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**Checklist**

- V-n diagram?
- Maneuvering envelope?
- Level turning flight?
- Limiting factors?
- Wind-up turn?
- Corner velocity?
**Herbst Maneuver**

- Minimum-time reversal of direction
- Kinetic-/potential-energy exchange
- Yaw maneuver at low airspeed
- X-31 performing the maneuver

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**Next Time:**

**Aircraft Equations of Motion**

*Reading:*

*Flight Dynamics,*

Section 3.1, 3.2, pp. 155-161

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**Learning Objectives**

What use are the equations of motion?
How is the angular orientation of the airplane described?
What is a cross-product-equivalent matrix?
What is angular momentum?
How are the inertial properties of the airplane described?
How is the rate of change of angular momentum calculated?
**Supplemental Material**

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**Gliding Flight of the P-51 Mustang**

**Maximum Range Glide**

- Loaded Weight = 9,200 lb (3,465 kg)
- \((L/D)_{\text{max}} = \frac{1}{2.8C_{D_0}} = 16.31\)
- \(\gamma_{\text{max}} = -\cot^{-1}\left(\frac{L}{D}\right)_{\text{max}} = -\cot^{-1}(16.3) = -3.5^\circ\)
- \((C_D)_{\text{L,max}} = 2C_{D_0} = 0.0326\)
- \((C_L)_{\text{L,max}} = \sqrt{\frac{C_{D_0}}{\epsilon}} = 0.531\)
- \(V_{L,max} = 76.49 \text{ m/s}\)
- \(h_{L,max} = V \sin \gamma = 4.68 \text{ m/s}\)
- \(R_{\text{L,glide}} = (16.31)(10) = 163.1 \text{ km}\)

**Maximum Endurance Glide**

- Loaded Weight = 9,200 lb (3,465 kg)
- \(S = 21.83 \text{ m}^2\)
- \(C_{D_{\text{max}}} = 4C_{D_0} = 4(0.0163) = 0.0652\)
- \(C_{L_{\text{max}}} = \sqrt{\frac{3C_{D_{\text{max}}}}{\epsilon}} = \sqrt{\frac{3(0.0163)}{0.0576}} = 0.921\)
- \((L/D)_{\text{end}} = 14.13\)
- \(h_{\text{end}} = -\frac{2W}{\rho \sqrt{S}} \left(\frac{C_{D_{\text{max}}}}{C_{L_{\text{max}}}}\right) = -\frac{4.11}{\sqrt{\rho}} \text{ m/s}\)
- \(\gamma_{\text{end}} = -4.05^\circ\)
- \(V_{\text{end}} = \frac{58.12}{\sqrt{\rho}} \text{ m/s}\)