Propagation of Uncertainty in Dynamic Systems

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- Propagation of the mean and variance in linear, time-varying discrete-time and continuous-time systems
- Markov processes and the transition function property
- Sampled-data representation of continuous-time systems
- White and colored noise inputs

Waves

- Deterministic and random components
- After a few seconds, the underlying model can be perceived
- But specifics of future waves cannot be predicted without random error
Linear, Time-Varying (LTV) Dynamic Model

- Discrete-time LTV model with known coefficients

\[ x_k = \Phi_{k-1} x_{k-1} + \Gamma_{k-1} u_{k-1} + \Lambda_{k-1} w_{k-1} \]

- Random initial condition and disturbance inputs
- All random variables assumed to be Gaussian, i.e., they are fully described by means and covariances

Statistics of Random Variables

Gaussian uncertainty model for state mean and covariance

\[ \bar{x}_0 = E[x_0]; \quad P_0 = E\{[x_0 - \bar{x}_0][x_0 - \bar{x}_0]^T\} \]
\[ \bar{x}_k = E[x_k]; \quad P_k = E\{[x_k - \bar{x}_k][x_k - \bar{x}_k]^T\} \]

Disturbance is assumed to have zero mean

\[ \bar{w}_k = 0; \quad Q_k = E\{[w_k][w_k]^T\} \]

Control is assumed to be known without error

\[ u_k = \bar{u}_k; \quad E[u_k - \bar{u}_k] \triangleq U_k = 0 \]
Multivariate Expected Values: State

Mean value vector of state vector

\[
\bar{x} = E(x) = \int_{-\infty}^{\infty} x \, pr(x) \, dx = \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \\ \vdots \\ \bar{x}_n \end{bmatrix}
\]

Covariance matrix of state vector

\[
P = E[(x - \bar{x})(x - \bar{x})^T] = \int_{-\infty}^{\infty} (x - \bar{x})(x - \bar{x})^T \, pr(x) \, dx
\]

Probability density function of the state

\[
pr(x) = \frac{1}{(2\pi)^{n/2} |P|^{1/2}} e^{-\frac{1}{2}(x-\bar{x})^T P^{-1}(x-\bar{x})}
\]

Covariance Matrix of State Vector

\[
P = E\left[ (x - \bar{x})(x - \bar{x})^T \right]
\]

\[
\begin{bmatrix}
\sigma_{x_1}^2 & \cdots & \rho_{1n} \sigma_{x_1} \sigma_{x_n} \\
\vdots & \ddots & \vdots \\
\rho_{1n} \sigma_{x_1} \sigma_{x_n} & \cdots & \sigma_{x_n}^2
\end{bmatrix}
\]

\[
\sigma_{x_i}^2 = \text{Variance of } x_i
\]

\[
\rho_{ij} = \text{Correlation coefficient for } x_i \text{ and } x_j, \text{ in } (-1, +1)
\]

\[
\rho_{12} \sigma_{x_1} \sigma_{x_2} = \text{Covariance of } x_1 \text{ and } x_2
\]

Gaussian probability distribution is completely described by its mean value and covariance matrix
Initial Condition and Control of the LTV Dynamic Model

Initial condition described by mean and covariance

\[
E(x_0) = \bar{x}_0 \triangleq m_o ; \quad E\left[(x_0 - m_o)(x_0 - m_o)^T\right] \triangleq P_o
\]

Control input is known precisely

\[
E[u_k] = \bar{u}_k = u_k ; \quad E\left[(u_k - \bar{u}_k)(u_k - \bar{u}_k)^T\right] = U_k = 0
\]

Cross-covariances are zero

\[
E\left[(x_k - \bar{x}_k)w_k^T\right] = M_k = 0 \\
E\left[(x_k - \bar{x}_k)u_k^T\right] = 0 \\
E\left[w_k u_k^T\right] = 0
\]

Probability Density Function of the LTV Dynamic Model

Initial probability density function depends only on the mean and covariance of the Gaussian distribution

\[
pr(x_0) = \frac{1}{(2\pi)^{n/2} \left| P_0 \right|^{1/2}} e^{-\frac{1}{2} (x_0 - m_0)^T P_o^{-1} (x_0 - m_0)}
\]
Multivariate Expected Values: Disturbance

Mean value vector of disturbance vector

\[ \bar{w} = E(w) = \int_{-\infty}^{\infty} w \, pr(w) \, dw = \begin{bmatrix} \bar{w}_1 \\ \bar{w}_2 \\ \vdots \\ \bar{w}_s \end{bmatrix} \]

Covariance matrix of disturbance vector

\[ Q \triangleq E \left[ (w - \bar{w}) (w - \bar{w})^T \right] = \int_{-\infty}^{\infty} (w - \bar{w}) (w - \bar{w})^T \, pr(w) \, dw \]

Probability density function of the disturbance

\[ pr(w) = \frac{1}{(2\pi)^{s/2} |Q|^{1/2}} e^{-\frac{1}{2} (w - \bar{w}) Q^{-1} (w - \bar{w})} \]

Covariance Matrix of Disturbance Vector

\[ Q = E \left[ (w - \bar{w}) (w - \bar{w})^T \right] \]

\[ \begin{bmatrix} \sigma_{w_1}^2 & \ldots & \rho_{1,s} \sigma_{w_1} \sigma_{w_s} \\ \ldots & \ldots & \ldots \\ \rho_{1,s} \sigma_{w_1} \sigma_{w_s} & \ldots & \sigma_{w_s}^2 \end{bmatrix} \]

- \( \sigma_{w_i}^2 \) = Variance of \( w_i \)
- \( \rho_{12} \) = Correlation coefficient for \( w_1 \) and \( w_2 \), in \((-1,+1)\)
- \( \rho_{12} \sigma_{w_i} \sigma_{w_2} \) = Covariance of \( w_1 \) and \( w_2 \)
Disturbance Input of the LTV Dynamic Model

**Disturbance input** (‘process noise’) is a white-noise sequence

\[
E(w_k) = 0 \\
E(w_k w_k^T) = Q'_k; \quad E(w_k w_{k-l}^T) = 0, \quad l = \text{any non-zero integer} \\
\text{or} \\
E(w_j w_{k}^T) = Q'_k \delta_{jk}
\]

\[
\delta_{jk} \triangleq \text{Kronecker delta function} = \begin{cases} 
1, & j = k \\
0, & j \neq k
\end{cases}
\]

Propagation of the Ensemble Mean Value Estimate

Expected value of the state

\[
E(\bar{x}_{k+1}) = E(\Phi \bar{x}_k + \Gamma u_k + \Lambda w_k)
\]

\[
\bar{x}_{k+1} = \Phi \bar{x}_k + \Gamma u_k + 0, \quad \bar{x}_0 \text{ given}
\]
Propagation of the Ensemble State Covariance Estimate

State covariance matrix

Expected values of cross terms are zero

\[
P_{k+1} = E \left[ (x_{k+1} - \bar{x}_{k+1})(x_{k+1} - \bar{x}_{k+1})^T \right]
\]

\[
P_{k+1} = E \left[ \Phi(x_k - \bar{x}_k) + \Gamma(u_k - \bar{u}_k) \right] E \left[ \Phi(x_k - \bar{x}_k) + \Gamma(u_k - \bar{u}_k) \right]^T
\]

\[
P_{k+1} = E \left[ \Phi(x_k - \bar{x}_k)(x_k - \bar{x}_k)^T \right] \Phi^T + 0 + \Lambda w_k w_k^T \Lambda^T
\]

\[
= \Phi E \left[ (x_k - \bar{x}_k)(x_k - \bar{x}_k)^T \right] \Phi^T + \Lambda E(w_k w_k^T) \Lambda^T
\]

\[
= \Phi P_k \Phi^T + \Lambda Q_k \Lambda^T, \quad P_0 \text{ given}
\]

Propagation of the Estimated State Probability Density Function

Uncertainty propagation model is a Markov Process

\[
\bar{x}_{k+1} = \Phi \bar{x}_k + \Gamma \bar{u}_k, \quad \bar{x}_0 \text{ given}
\]

\[
P_{k+1} = \Phi P_k \Phi^T + \Lambda Q_k \Lambda^T, \quad P_0 \text{ given}
\]

Propagating the state mean and covariance is equivalent to propagating the entire probability density function of the state

\[
pr(x_{k+1}) = \frac{1}{(2\pi)^{n/2} |P_{k+1}|^{1/2}} e^{-\frac{1}{2}(x_{k+1} - \bar{x}_{k+1})^T P_{k+1}^{-1}(x_{k+1} - \bar{x}_{k+1})}
\]
Expected Value of the State

First moment of $x$

$$E(x_k) = E\left( \Phi_{k-1} x_{k-1} + \Gamma_{k-1} u_{k-1} + \Lambda_{k-1} w_{k-1} \right)$$

$$\bar{x}_k = \Phi_{k-1} \bar{x}_{k-1} + \Gamma_{k-1} \bar{u}_{k-1} + \Lambda_{k-1} (0)$$

$$m_k = \Phi_{k-1} m_{k-1} + \Gamma_{k-1} u_{k-1}$$

Expected Value of the State Covariance

Second central moment of $x$

$$P_k \triangleq E \left[ (x_i - m_i)(x_i - m_i)^T \right] = E \left[ \Phi_{k-1} (x_{k-1} - m_{k-1}) + \Gamma_{k-1} (u_{k-1} - \bar{u}_{k-1}) + \Lambda_{k-1} (w_{k-1} - \bar{w}_{k-1}) \right]$$

With negligible cross-covariance

$$P_k = \Phi_{k-1} E \left[ (x_{k-1} - m_{k-1})(x_{k-1} - m_{k-1})^T \right] \Phi_{k-1}^T + \Lambda_{k-1} E \left[ (w_{k-1} - \bar{w}_{k-1})(w_{k-1} - \bar{w}_{k-1})^T \right] \Lambda_{k-1}^T$$

$$= \Phi_{k-1} P_{k-1} \Phi_{k-1}^T + \Lambda_{k-1} Q_{k-1} \Lambda_{k-1}^T$$

$$\triangleq \Phi_{k-1} P_{k-1} \Phi_{k-1}^T + Q_{k-1}$$

$$Q_{k-1} \triangleq \Lambda_{k-1} Q'_{k-1} \Lambda_{k-1}^T$$

$$\dim(Q_{k-1}) = n \times n$$

$$\dim(Q'_{k-1}) = s \times s$$
Probability Mass and Density Functions

The random sequence - has the transition function property, i.e., current value conditioned on prior value - is a Gauss-Markov sequence

\[ \Delta x \xrightarrow{\Delta x_{k-1}} \Delta x_{k-2} \xrightarrow{\Delta x_{k-3}} \cdots \xrightarrow{\Delta x_0} \]

\[ P(\Delta x_k) = \prod_{i=1}^{k} \Pr(\Delta x_i | \Delta x_{i-1}) \Pr(\Delta x_0) \]

Conditional Probability Density Function of the State

\[ p(x_k) = \frac{1}{\sqrt{2\pi \sigma^2}} e^{-\frac{1}{2}(x_k - \mu)^2} \]

\[ m_k = \Phi_{k-1}m_{k-1} + \Gamma_{k-1}u_{k-1} \]

\[ P_k = \Phi_{k-1}P_{k-1}\Phi_{k-1}^T + Q_{k-1} \]

The density function is conditioned on the prior state

\[ p(x_k | x_{k-1}) = \frac{1}{\sqrt{2\pi \sigma^2}} e^{-\frac{1}{2}(x_k - \mu)^2} \]

... and propagation is a Markov process

\[ p(x_k) = p(x_k | x_{k-1})p(x_{k-1}) = p(x_k | x_{k-1}) \text{ if } p(x_{k-1}) = 1 \]
Gauss-Markov Sequence

\[
\begin{align*}
m_k &= \Phi_{k-1} m_{k-1} + \Gamma_{k-1} u_{k-1} \\
P_k &= \Phi_{k-1} P_{k-1} \Phi_{k-1}^T + Q_{k-1}
\end{align*}
\]

- The mean and covariance are completely specified by the prior probability distribution
- The random sequence has the transition function property, i.e., current value (pdf) conditioned on prior pdf
- is a Gauss-Markov sequence

\[
\begin{align*}
\text{pr}(x_k) &= \text{pr}(x_k | x_{k-1}) \text{pr}(x_{k-1}) = \text{pr}(x_k | x_{k-1}) \text{pr}(x_{k-1} | x_{k-2}) \text{pr}(x_{k-2}) = \cdots \\
&= \left[ \prod_{i=1}^{k} \text{pr}(x_i | x_{i-1}) \right] \text{pr}(x_0)
\end{align*}
\]
Sampled-Data Representation of Continuous-Time Systems

Continuous-time LTV model with known coefficients

\[ \dot{x}(t) = F(t)x(t) + G(t)u(t) + L(t)w(t), \quad x(t_0) \text{ given} \]

\[ x(t) = x(t_0) + \int_{t_0}^{t} \left[ F(\tau)x(\tau) + G(\tau)u(\tau) + L(\tau)w(\tau) \right] d\tau \]

Incremental solution

\[ x(t_k) = x(t_{k-1}) + \int_{t_{k-1}}^{t_k} \left[ Fx(\tau) + Gu(\tau) + Lw(\tau) \right] d\tau \]

\[ = \Phi(t_k, t_{k-1})x(t_{k-1}) + \int_{t_{k-1}}^{t_k} \Phi(t_k, \tau)[G(\tau)u(\tau) + L(\tau)w(\tau)] d\tau \]

Descriptions of Random Variables

\[ E(x_0) = m_o; \quad E[ (x_0 - m_o)(x_0 - m_o)^T ] = P_0 \]

\[ E[w(t)] = 0 \]

\[ E[w(t)w^T(\tau)] = Q'\delta(t-\tau) \]

\[ E[u(t)] = u(t); \quad E\left[ [u(t) - \bar{u}(t)][u(t) - \bar{u}(t)]^T \right] = 0 \]
Mean Value and Covariance Solutions

Mean value propagation from $t_{k-1}$ to $t_k$

$$m_k = \Phi_k m_{k-1} + \int_{t_{k-1}}^{t_k} \Phi(t_k, \tau) G(\tau) u(\tau) d\tau$$

Covariance propagation

Double integration over time ($\tau$ and $\alpha$)

$$P_k = \Phi_{k-1} P_{k-1} \Phi_{k-1}^T + E \left\{ \int_{t_{k-1}}^{t_k} \Phi(t_k, \tau) L(\tau) w(\tau) d\tau \int_{t_{k-1}}^{t_k} \Phi(t_k, \alpha) L(\alpha) w(\alpha) d\alpha \right\}$$

$$= \Phi_{k-1} P_{k-1} \Phi_{k-1}^T + \int_{t_{k-1}}^{t_k} \int_{t_{k-1}}^{t_k} \Phi(t_k, \tau) L(\tau) E \left[ w(\tau) w^T(\alpha) \right] L^T(\alpha) \Phi^T(t_k, \alpha) d\tau d\alpha$$

Covariance Propagation

$$E \left[ w(\tau) w^T(\alpha) \right] \triangleq Q_C \delta(\tau - \alpha)$$

$$P_k = \Phi_{k-1} P_{k-1} \Phi_{k-1}^T + \int_{t_{k-1}}^{t_k} \int_{t_{k-1}}^{t_k} \Phi(t_k, \tau) L(\tau) Q_C \delta(\tau - \alpha) L^T(\alpha) \Phi^T(t_k, \alpha) d\tau d\alpha$$

$$= \Phi_{k-1} P_{k-1} \Phi_{k-1}^T + \int_{t_{k-1}}^{t_k} \Phi(t_k, \tau) L(\tau) Q_C L^T(\tau) \Phi^T(t_k, \tau) d\tau$$

$$\triangleq \Phi_{k-1} P_{k-1} \Phi_{k-1}^T + Q_{k-1}$$
Relationship Between Discrete- and Continuous-Time Disturbance Models

\[ Q_{k-1} = \int_{t_{k-1}}^{t_k} \Phi(t_k, \tau) L(\tau) Q_C' L^T(\tau) \Phi^T(t_k, \tau) d\tau \]

\[ Q_{k-1} : \text{ Covariance matrix } (n \times n) \]
\[ Q_C' : \text{ Spectral density matrix } (s \times s) \]

For small \( \Delta t \)

\[ Q_{k-1} \approx L(t_{k-1}) Q_C'(t_{k-1}) L^T(t_{k-1}) \Delta t \]
\[ \neq L(t_{k-1}) Q_C'(t_{k-1}) L^T(t_{k-1}) \Delta t^2 \]

Autocovariance Functions for Continuous-and Discrete-Time White Noise \((Scalar)\)

\[ \phi_{ww}(\Delta t) = Q_C' \delta(\Delta t) \]
\[ \phi_{ww}(\Delta t_k) = Q_{k-1} \delta(\Delta t_k) \]

\[ \delta(\Delta t) = \begin{cases} 
\infty, & \Delta t = 0 \\
0, & \Delta t \neq 0 
\end{cases} \]
\[ \int_{-\infty}^{\infty} \delta(t) dt = \lim_{\Delta t \to 0} \int_{-\Delta t}^{\Delta t} \delta(t) d(t) = 1 \]

\[ \delta_{jk} = \delta(\Delta t_k) = \begin{cases} 
1, & j = k \\
0, & j \neq k 
\end{cases} \]
Colored Noise, Variance, Spectral Density Matrices (Scalar)

- Spreading of the autocovariance function is accompanied by
  - Finite variance \( \phi_x(\tau) \)
  - Low-pass filtering of the power spectral density

\[
\Phi_{xx}(\omega) = \int_{-\infty}^{\infty} \phi_x(\tau) e^{-j\omega \tau} d\tau
\]

**Autocovariance Function**

**Power Spectral Density Function**

Spectral Density Matrices

- Dynamic systems subject all white-noise inputs to low-pass filtering
- Nyquist (or folding) frequency

\[
\omega_{\text{Nyquist}} = \frac{1}{2} \omega_{\text{Sampling}} = \frac{\pi}{T}
\]

\( T = \) Sampling interval, sec

- Signals above this frequency fold and corrupt lower-frequency sampled signals (aliasing)

**Caveat:** Covariance and Spectra capture “stochastic equilibrium” effects but not transient effects
Frequency Folding (Aliasing)

Quantization, Zero-Order Hold, and Inter-Sample Ripple

- Digital control subject to zero-order hold in D/A
- Sampled signal misses inter-sample transient response
- Effective delay of sampled signal
- Continuous signal sampled with finite precision in A/D
- Inter-sample ripple
Colored Noise Disturbance

- Disturbance may not be “white” noise
  - Power may vary with frequency

- Mean
  \[ E(w_k) = 0 \]

- Covariance
  \[ E(w_k w_k^T) = W_k \]
  \[ E(w_k w_{k-1}^T) \neq 0 = V_k \]

- Linear model for disturbance propagation
  - Driven by white noise sequence
  \[ w_k = A_{k-1} w_{k-1} + \eta_{k-1} \]

- Correlation with adjacent signal

Calculation of Colored Disturbance Process Parameters

Autocovariance of scalar Markov process

\[ \phi_x(i) = E(x_i x_{i+1}) = bE(x_i^2) = b\sigma_i^2 \]
\[ \phi_x(k) = b^k \sigma_i^2 = b^k \sigma^2 \]

Autocovariance of vector Markov process

\[ E(w_k w_{k-1}^T) = E[(A_{k-1} w_{k-1} + \eta_{k-1}) w_{k-1}^T] \]
\[ = A_{k-1} E(w_{k-1} w_{k-1}^T) \]
\[ = A_{k-1} W_{k-1} \]
\[ = V_k \]

Therefore, state transition matrix of the noise model is

\[ A_{k-1} = V_k W_{k-1}^{-1} \]
Propagating the Colored Disturbance

\[ W_k = A_{k-1} W_{k-1} A_{k-1}^T + Q_{\eta_k} \]

- System state vector is augmented to include the disturbance
- Augmented equation for the mean
  \[
  \begin{bmatrix}
  m_k \\
  w_k
  \end{bmatrix} =
  \begin{bmatrix}
  \Phi_{k-1} & \Lambda_{k-1} \\
  0 & A_{k-1}
  \end{bmatrix}
  \begin{bmatrix}
  m_{k-1} \\
  w_{k-1}
  \end{bmatrix} +
  \begin{bmatrix}
  \Gamma_{k-1} \\
  0
  \end{bmatrix}
  u_{k-1} +
  \begin{bmatrix}
  0 \\
  I_s
  \end{bmatrix}
  \eta_{k-1}
  \]

- Augmented covariance equation
  \[
  \begin{bmatrix}
  P_k \\
  0 \\
  0 \\
  W_k
  \end{bmatrix} =
  \begin{bmatrix}
  \Phi_{k-1} & \Lambda_{k-1} \\
  0 & A_{k-1}
  \end{bmatrix}
  \begin{bmatrix}
  P_{k-1} \\
  0 \\
  0 \\
  W_{k-1}
  \end{bmatrix} +
  \begin{bmatrix}
  \Phi_{k-1} & \Lambda_{k-1} \\
  0 & A_{k-1}
  \end{bmatrix}^T
  \begin{bmatrix}
  0 \\
  I_s
  \end{bmatrix}
  \cdot
  \begin{bmatrix}
  Q_{\eta_{k-1}} & 0 \\
  0 & I_s
  \end{bmatrix}
  \]

Next Time:
Kalman Filter for Discrete-Time Systems
Covariance Matrix is Expected Value of the Outer Product

- Transcript correlation for normal and abnormal samples
  - Identifies possible components of biological circuits

- Sample tissue correlation for RNA expression
  - Confirms or questions the classification of samples
Example: Propagating a Scalar Probability Density Function

Scalar LTI system with zero-mean Gaussian random input

\[ x_{k+1} = bx_k + \sqrt{1 - b^2} u_k + \sqrt{1 - b^2} w_k, \quad x_0 \text{ given} \]

Propagation of the mean value

\[ \bar{x}_{i+1} = b\bar{x}_i + \sqrt{1 - b^2} \bar{u}_i, \quad \bar{x}_0 \text{ given} \]

Propagation of the variance

\[ P_{k+1} = b^2 P_k + (1 - b^2)Q_k, \quad P_0 \text{ given} \]

\[ Q_k = E(w_k^2) \]

Example: Propagating a Scalar Probability Density Function

\[ x_0 = 0; \quad P_0 = 0 \]

Propagation of Uncertainty, \( b = 0.99, Q = 1 \)
Example: Propagating a Scalar Probability Density Function

\[ x_0 = 1; \quad P_0 = 0 \]

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Example: Propagating a Scalar Probability Density Function

\[ x_0 = 0; \quad P_0 = 3 \]
Gaussian PDF Propagation Program

Scalar Propagation of Mean and Variance
Rob Stengel
4/5/2011

clear

% Propagation of Uncertainty

date
b = 0.99; b2 = b*b; bb = 1 - b^2; sqrb = sqrt(bb)

w = []; x = []; xbar = [];
x(1) = 0; xbar(1) = x(1)

P(1) = 1
Q = 1
u = 0

for k = 1:999
    w(k) = randn(1);
    x(k+1) = b*x(k) + sqrb*u + sqrb*w(k);
    xbar(k+1) = b*xbar(k) + sqrb*u;
    P(k+1) = b2*P(k) + bb*Q;
end

k = 1:1000;
figure
plot(k,x,'b',k,xbar,'r',k,(xbar+sqrt(P)),'k',k,(xbar-sqrt(P)),'k',k,w,'c'),grid

legend('State', 'Mean', 'Mean + Sigma', 'Mean - Sigma', 'Disturbance')