The Problem: Minimize Cost, Subject to Dynamic Constraint, Uncertain Disturbances, and Measurement Error

\[ \dot{x}(t) = Fx(t) + Gu(t) + Lw(t), \quad x(0) = x_0 \]

Cost Function

\[ \min V(t_o) = \min_u J(t_f) \]

\[ J_D(t) = \left\{ \dot{x}(t), P(t), u(t) \right\} \]
Initial Conditions and Dimensions

\[ E[\mathbf{x}(0)] = \hat{\mathbf{x}}_o; \quad E\left\{ [\mathbf{x}(0) - \hat{\mathbf{x}}(0)][\mathbf{x}(0) - \hat{\mathbf{x}}(0)^T] \right\} = \mathbf{P}(0) \]

\[ E[\mathbf{w}(t)] = 0; \quad E[\mathbf{w}(t)\mathbf{w}^T(\tau)] = \mathbf{W}\delta(t - \tau) \]

\[ E[\mathbf{n}(t)] = 0; \quad E[\mathbf{n}(t)\mathbf{n}^T(\tau)] = \mathbf{N}\delta(t - \tau) \]

\[ E[\mathbf{w}(t)\mathbf{n}^T(\tau)] = 0 \]

\[ \text{dim}[\mathbf{x}(t)] = n \times 1 \]

\[ \text{dim}[\mathbf{u}(t)] = m \times 1 \]

\[ \text{dim}[\mathbf{w}(t)] = s \times 1 \]

\[ \text{dim}[\mathbf{z}(t)] = \text{dim}[\mathbf{n}(t)] = r \times 1 \]

Linear-Quadratic-Gaussian Control
Separation Property and Certainty Equivalence

- **Separation Property**
  - Optimal Control Law and Optimal Estimation Law can be derived separately
  - Their derivations are strictly independent

- **Certainty Equivalence Property**
  - Separation property plus, ...
  - The Stochastic Optimal Control Law and the Deterministic Optimal Control Law are the same
  - The Optimal Estimation Law can be derived separately

- **Linear-quadratic-Gaussian (LQG) control is certainty-equivalent**

### The Equations

**Continuous-Time Model**

- **System State and Measurement**
  \[ \dot{x}(t) = F(t)x(t) + G(t)u(t) + L(t)w(t) \]
  \[ z(t) = Hx(t) + n(t) \]

- **Control Law**
  \[ u(t) = -C(t)\hat{x}(t) + C_F(t)y_C(t) \]

- **State Estimate**
  \[ \dot{\hat{x}}(t) = F(t)\hat{x}(t) + G(t)u(t) + K(t)[z(t) - H\hat{x}(t)] \]
  \[ = [F(t) - G(t)C(t) - K(t)H]\hat{x}(t) + G(t)C_F(t)y_C(t) + K(t)z(t) \]

- **Estimator Covariance Gain and State Covariance Estimate**
  \[ K(t) = \hat{P}(t)H^TN^{-1}(t) \]
  \[ \dot{\hat{P}}(t) = F(t)\hat{P}(t) + \hat{P}(t)F^T(t) + L(t)W(t)L^T(t) - \hat{P}(t)H^TN^{-1}(t)HP(t) \]

- **Control Gain and Adjoint Covariance Estimate**
  \[ C(t) = R^{-1}(t)G^T(t)S(t) \]
  \[ \dot{S}(t) = -Q(t) - F(t)^T S(t) - S(t)F(t) + S(t)G(t)R^{-1}(t)G^T(t)S(t) \]
Estimator in the Feedback Loop

Linear-Gaussian (LG) state estimator adds dynamics to the feedback signal

\[ u(t) = -C(t)\hat{x}(t) \]

\[ \dot{x}(t) = F\hat{x}(t) + Gu(t) + K(t)[z(t) - H\hat{x}(t)] \]

Thus, state estimator can be viewed as a control-law “compensator”

Bandwidth of the compensation is dependent on the multivariable signal/noise ratio, \([PH^T]N^{-1}\)

\[ K(t) = P(t)H^T N^{-1} \]

\[ \dot{P}(t) = FP(t) + P(t)F^T + LWL^T - P(t)H^T N^{-1} HP(t) \]

Stable Scalar LTI Example of Estimator Compensation

Dynamic system (stable) and measurement

\[ \dot{x} = -x + w; \quad z = Hx + n \]

Estimator differential equation

\[ \dot{\hat{x}} = -\hat{x} + K(z - H\hat{x}) = (-1 - KH)\hat{x} + Kz \]

Laplace transform of estimator

\[ [s - (-1 - KH)]\hat{x}(s) = Kz(s) \]

Estimator transfer function

Low-pass filter

\[ \frac{\hat{x}(s)}{z(s)} = \frac{K}{s - (-1 - KH)} \]
Unstable Scalar LTI Example of Estimator Compensation

Dynamic system (unstable) and measurement
\[ \dot{x} = x + w; \quad z = Hx + n \]

Estimator differential equation
\[ \dot{\hat{x}} = \hat{x} + z - H\hat{x} = (1 - KH)\hat{x} + Kz \]

Laplace transform of estimator
\[ [s - (1 - KH)]\hat{x}(s) = Kz(s) \]

Estimator transfer function
Low-pass filter
\[ \hat{x}(s) = \frac{K}{s - (1 - KH)}z(s) \]

Steady-State Scalar Filter Gain

\[ K = \frac{PH}{N} \]

\[ 0 = 2P + W - \frac{P^2H^2}{N}; \quad P^2 - \frac{2N}{H^2} - \frac{WN}{H^2} = 0 \]

Algebraic Riccati equation (unstable case)
\[ P = \frac{N}{H^2} \pm \sqrt{\left( \frac{N}{H^2} \right)^2 + \frac{WN}{H^2}} = \frac{N}{H^2} \left[ 1 \mp \frac{WH^2}{N} \right] \]
Steady-State Filter Gain

\[ K = \left\{ \frac{N}{H^2} \left[ 1 + \sqrt{1 + \frac{WH^2}{N}} \right] \right\} H = \left\{ \frac{1}{H} \left[ 1 + \sqrt{1 + \frac{WH^2}{N}} \right] \right\} \]

\[ \frac{K}{W \gg N} \rightarrow \sqrt{\frac{W}{N}} \]

Dynamic Constraint on the Certainty-Equivalent Cost

\( P(t) \) is independent of \( u(t) \); therefore

\[ \min_u J = \min_u J_{CE} + J_S \]

\( J_{CE} \) is

Identical in form to the deterministic cost function

Minimized subject to dynamic constraint based on the state estimate

\[ \hat{x}(t) = F\hat{x}(t) + Gu(t) + K(t)\left[ z(t) - H\hat{x}(t) \right] \]
Kalman-Bucy Filter Provides Estimate of the State Mean Value

Filter residual is a Gaussian process

\[
\hat{x}(t) = F\hat{x}(t) + Gu(t) + K(t)\left[z(t) - H\hat{x}(t)\right]
\]

\[
\triangleq F\hat{x}(t) + Gu(t) + K(t)\varepsilon(t)
\]

Filter equation is analogous to deterministic dynamic constraint on deterministic cost function

\[
\dot{x}(t) = Fx(t) + Gu(t) + Lw(t)
\]

Control That Minimizes the Certainty-Equivalent Cost

Optimizing control history is generated by a time-varying feedback control law

\[
u(t) = -C(t)\hat{x}(t)
\]

The control gain is the same as the deterministic gain

\[
C(t) = R^{-1}G^TS(t)
\]

\[
\dot{S}(t) = -Q - F^TS(t)S(t)F + S(t)GR^{-1}G^TS(t)
\]

\[
S(t_f) \text{ given}
\]
Optimal Cost for the Continuous-Time LQG Controller

Certainty-equivalent cost

\[ J_{CE} = \frac{1}{2} \text{Tr} \left[ S(0)E\left[ \hat{x}(0)\hat{x}(0)^T \right] + \int_0^{t_f} S(t)K(t)NK^T(t)dt \right] \]

Total cost

\[ J = J_{CE} + J_s = \frac{1}{2} \text{Tr} \left[ S(0)E\left[ \hat{x}(0)\hat{x}(0)^T \right] + \int_0^{t_f} S(t)K(t)NK^T(t)dt \right] + \frac{1}{2} \text{Tr} \left[ S(t_f)P(t_f) + \int_0^{t_f} Q(t)P(t)dt \right] \]

Discrete-Time LQG Controller

Kalman filter produces state estimate

\[ \hat{x}_k(-) = \Phi \hat{x}_{k-1}(+) + \Gamma C_{k-1} \hat{x}_{k-1}(+) \]

\[ \hat{x}_k(+) = \hat{x}_k(-) + K_k \left[ z_k - H \hat{x}_k(-) \right] \]

Closed-loop system uses state estimate for feedback control

\[ x_{k+1} = \Phi x_k - \Gamma C_1 \hat{x}_k(+) \]
Response of Discrete-Time 1st-Order System to Disturbance
Kalman Filter Estimate from Noisy Measurement

Comparison of 1st-Order Discrete-Time LQ and LQG Control Response

Linear-Quadratic Control with Noise-free Measurement
Linear-Quadratic-Gaussian Control with Noisy Measurement
Asymptotic Stability of the LQG Regulator

System Equations with LQG Control

With perfect knowledge of the system

\[ \dot{x}(t) = Fx(t) + Gu(t) + Lw(t) \]
\[ \dot{\hat{x}}(t) = F\hat{x}(t) + Gu(t) + K(t)[z(t) - H\hat{x}(t)] \]

State estimate error

\[ \varepsilon(t) = x(t) - \hat{x}(t) \]

State estimate error dynamics

\[ \dot{\varepsilon}(t) = (F - KH)\varepsilon(t) + Lw(t) - Kn(t) \]
Control-Loop and Estimator
Eigenvalues are Uncoupled

\[
\begin{bmatrix}
\dot{x}(t) \\
\dot{\varepsilon}(t)
\end{bmatrix} =
\begin{bmatrix}
(F - GC) & GC \\
0 & (F - KH)
\end{bmatrix}
\begin{bmatrix}
x(t) \\
\varepsilon(t)
\end{bmatrix} +
\begin{bmatrix}
L & 0 \\
L & -K
\end{bmatrix}
\begin{bmatrix}
w(t) \\
n(t)
\end{bmatrix}
\]

Upper-block-triangular stability matrix
LQG system is stable because
(F – GC) is stable
(F – KH) is stable

Estimate error affects state response
Actual state does not affect error response
Disturbance affects both equally

Parameter Uncertainty
Introduces Coupling
Coupling Due To Parameter Uncertainty

Actual System: \( \{ F_A, G_A, H_A \} \)
Assumed System: \( \{ F, G, H \} \)

\[
\begin{align*}
\dot{x}(t) &= F_A x(t) + G_A u(t) + Lw(t) \\
\dot{\hat{x}}(t) &= F \hat{x}(t) + Gu(t) + K(t) [ z(t) - H \hat{x}(t) ] \\
\hat{y}(t) &= H_A x(t) + n(t) \\
u(t) &= -C(t) \hat{x}(t)
\end{align*}
\]

Closed-loop control and estimator responses are coupled

Effects of Parameter Uncertainty on Closed-Loop Stability

\[
\begin{align*}
|sI_{2n} - F_{CL}| &= \\
&= \begin{vmatrix}
\begin{bmatrix}
|I_{n} - (F_A - G_A C) | \\
|G_A C |
\end{bmatrix} - [F + (G_A - G) C - KH] \\
|F_A - F - (G_A - G) C - K(H_A - H) |
\end{vmatrix}
\end{align*}
\]

\[= \Delta_{CL}(s) = 0\]

- Uncertain parameters affect closed-loop eigenvalues
- Coupling can lead to instability for numerous reasons
  - Improper control gain
  - Control effect on estimator block
  - Redistribution of damping
Doyle’s Counter-Example of LQG Robustness (1978)

\[
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2
\end{bmatrix} = \begin{bmatrix}
1 & 1 \\
0 & 1
\end{bmatrix} \begin{bmatrix}
x_1 \\
x_2
\end{bmatrix} + \begin{bmatrix}
0 \\
1
\end{bmatrix} u + \begin{bmatrix}
1 \\
1
\end{bmatrix} w
\]

\[
z = \begin{bmatrix}
1 & 0
\end{bmatrix} \begin{bmatrix}
x_1 \\
x_2
\end{bmatrix} + n
\]

Design Matrices

\[
Q = Q \begin{bmatrix}
1 & 1 \\
1 & 1
\end{bmatrix}; \quad R = 1; \quad W = W \begin{bmatrix}
1 & 1 \\
1 & 1
\end{bmatrix}; \quad N = 1
\]

Control and Estimator Gains

\[
C = \left(2 + \sqrt{4 + Q}\right) \begin{bmatrix}
1 & 1
\end{bmatrix} = \begin{bmatrix}
c & c
\end{bmatrix}
\]

\[
K = \left(2 + \sqrt{4 + W}\right) \begin{bmatrix}
1 & 1
\end{bmatrix} = \begin{bmatrix}
k & k
\end{bmatrix}
\]

Unstable Plant

Unstable Plant Design Matrices

Characteristic Equation

\[
\begin{vmatrix}
(s - 1) & -1 & 0 & 0 \\
0 & (s - 1) & \mu c & \mu c \\
-k & 0 & (s - 1 + k) & -1 \\
-k & 0 & (c + k) & (s - 1 + c)
\end{vmatrix} = 0
\]

\[
s^4 + a_3 s^3 + a_2 s^2 + a_1 s + a_0 = \Delta_{CL}(s) = 0
\]
Stability Effect of Parameter Variation

**Routh's Stability Criterion** (necessary condition)
- All coefficients of \( \Delta(s) \) must be positive for stability
  - \( \mu \) is nominally equal to 1
  - \( \mu \) can force \( a_0 \) and \( a_1 \) to change sign
  - Result is dependent on magnitude of \( ck \)

\[
\begin{align*}
a_1 &= k + c - 4 + 2(\mu - 1)ck \\
a_0 &= 1 + (1 - \mu)ck
\end{align*}
\]

- Arbitrarily small uncertainty, \( \mu = 1 + \epsilon \), could cause instability
- Not surprising: **uncertainty is in the control effect**

The Counter-Example Raises a Flag

**Solution**
Choose \( Q \) and \( W \) to be small, increasing allowable range of \( \mu \)

- However, .... The counter-example is irrelevant because it does not satisfy the requirements for LQ and LG stability
  - The open-loop system is **unstable**, so it requires feedback control to restore stability
  - To guarantee stability, \( Q \) and \( W \) must be positive definite, but

\[
\begin{align*}
Q &= Q \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}; \quad \text{hence}, |Q| = 0 \\
W &= W \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}; \quad \text{hence}, |W| = 0
\end{align*}
\]


**Restoring Robustness**  
*(Loop Transfer Recovery)*

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**Loop-Transfer Recovery**  
*(Doyle and Stein, 1979)*

- **Proposition:** LQG and LQ robustness would be the same if the control vector had the same effect on the state and its estimate $Cx(t)$ and $C\hat{x}(t)$ produce *same expected value of control*, $E[u(t)]$

  but not the same

  $$E\left\{ \left[ u_{LQ}(t) - u_{LQG}(t) \right] \left[ u_{LQ}(t) - u_{LQG}(t) \right]^T \right\}$$

  as $\hat{x}(t)$ contains measurement errors but $x(t)$ does not

- Therefore, restoring the *correct mean value* from $z(t)$ restores closed-loop robustness

- **Solution:** Increase the assumed “process noise” for estimator design as follows (see text for details)

  $$W = W_o + k^2GG^T$$

- Analogous solution in Kwakernaak, H., and Sivan, R.,  
  *Linear Optimal Control Systems*, 1972
Expression of Uncertainty in the System Model

System uncertainty may be expressed as
• Elements of $F$
• Coefficients of $\Delta(s)$
• Eigenvalues, $\lambda$
• Frequency response/singular values/time response, $A(j\omega), \sigma(j\omega), x(t)$

• Variation may be
  – Deterministic, e.g.,
    • Upper/lower bounds ("worst-case")
  – Probabilistic, e.g.,
    • Gaussian distribution

• Bounded variation is equivalent to probabilistic variation with uniform distribution
Stochastic Root Locus: 
Uncertain Damping Ratio and 
Natural Frequency

Laplace transform of dynamic model

\[ s^2 + 2\zeta\omega_n s + \omega_n^2 = 0 \]

Gaussian statistics

- \[ E(\zeta) = \bar{\zeta} = 0.707 \]
- \[ E(\omega_n) = \bar{\omega}_n = 1.0 \]

\[ E(\zeta - \bar{\zeta})^2 = 0.22 \]

\[ E((\omega_n - \bar{\omega}_n)^2) = 0.22 \]

Uniform Statistics

- \[ \zeta = [0.507, 0.907] \]
- \[ \omega_n = [0.8, 1.2] \]

Probability of Instability

- Nonlinear mapping from probability density functions (pdf) of uncertain parameters to pdf of roots
- Finite probability of instability with Gaussian (unbounded) distribution
- Zero probability of instability for some uniform distributions
Probabilistic Control Design

• Design constant-parameter controller (CPC) for satisfactory stability and performance in an uncertain environment

• Monte Carlo Evaluation of simulated system response with
  – competing CPC designs [Design parameters = d]
  – given statistical model of uncertainty in the plant [Uncertain plant parameters = v]

• Search for best CPC
  – Exhaustive search
  – Random search
  – Multivariate line search
  – Genetic algorithm
  – Simulated annealing

Design Outcome Follows Binomial Distribution

• Binomial distribution: Satisfactory/Unsatisfactory
• Confidence intervals of probability estimate are functions of
  – Actual probability
  – Number of trials

Maximum Information Entropy when \( Pr = 0.5 \)
Example: Probability of Stable Control of an Unstable Plant

Longitudinal dynamics for a Forward-Swept-Wing Airplane

\[ F = \begin{bmatrix} -2g f_{11} / V & \rho V^2 f_{12} / 2 & \rho V f_{13} & -g \\ -45 / V^2 & \rho V f_{22} / 2 & 1 & 0 \\ 0 & \rho V^2 f_{32} / 2 & \rho V f_{33} & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \]

\[ \mathbf{G} = \begin{bmatrix} g_{11} & g_{12} \\ 0 & 0 \\ g_{31} & g_{32} \\ 0 & 0 \end{bmatrix} \quad ; \quad \mathbf{x} = \begin{bmatrix} V, \text{Airspeed} \\ \alpha, \text{Angle of attack} \\ q, \text{Pitch rate} \\ \theta, \text{Pitch angle} \end{bmatrix} \]

Ray, Stengel, 1991

Example: Probability of Stable Control of an Unstable Plant

Nominal eigenvalues (one unstable)

\[ \lambda_{1-4} = -0.1 \pm 0.057 j, \quad -5.15, \quad 3.35 \]

Air density and airspeed, \( \rho \) and \( V \), have uniform distributions (±30%)

10 coefficients have Gaussian distributions (\( \sigma = 30\% \))

\[ \mathbf{p} = \begin{bmatrix} \rho & V & f_{11} & f_{12} & f_{13} & f_{22} & f_{32} & f_{33} & g_{11} & g_{12} & g_{31} & g_{32} \end{bmatrix}^T \]
LQ Regulators for the Example

Three stabilizing feedback control laws

Case a) LQR with low control weighting

\[ Q = \text{diag}(1,1,1,0); \quad R = (1,1); \quad \lambda_{1-4_{\text{nominal}}} = -35,-5.1,-3.3,-0.02 \]

\[ C = \begin{bmatrix} 0.17 & 130 & 33 & 0.36 \\ 0.98 & -11 & -3 & -1.1 \end{bmatrix} \]

Case b) LQR with high control weighting

\[ Q = \text{diag}(1,1,1,0); \quad R = (1000,1000); \quad \lambda_{1-4_{\text{nominal}}} = -5.2,-3.4,-1.1,-0.02 \]

\[ C = \begin{bmatrix} 0.03 & 83 & 21 & -0.06 \\ 0.01 & -63 & -16 & -1.9 \end{bmatrix} \]

Case c) Case b with gains multiplied by 5 for bandwidth (loop-transfer) recovery

\[ \lambda_{1-4_{\text{nominal}}} = -32,-5.2,-3.4,-0.01 \]

\[ C = \begin{bmatrix} 0.13 & 413 & 105 & -0.32 \\ 0.05 & -313 & -81 & -1.1 - 9.5 \end{bmatrix} \]

Stochastic Root Loci

- Distribution of closed-loop roots with
  - Gaussian uncertainty in 10 parameters
  - Uniform uncertainty in velocity and air density
- 25,000 Monte Carlo evaluations
- Probability of instability
  - a) Pr = 0.072
  - b) Pr = 0.021
  - c) Pr = 0.0076

Ray, Stengel, 1991
Probabilities of Instability for the Three Cases

95% CONFIDENCE INTERVALS (with dynamic pressure effects)

- Case a: Low LQ Control Weights
  - Probabilities of instability with 30% uniform aerodynamic uncertainty
    - Case a: $3.4 \times 10^{-4}$
    - Case b: 0
    - Case c: 0

- Case b: High LQ Control Weights
  - Probabilities of instability with Gaussian Aerodynamic Uncertainty
  - Case (b) $\phi = 0.0050$ $(L,U) = 0.01872, 0.0223$

- Case c: Bandwidth Recovery
  - Probabilities of instability with Gaussian Aerodynamic Uncertainty
  - Case (c) $\phi = 0.00786$ $(L,U) = 0.00609, 0.00809$
American Control Conference
Benchmark Control Problem, 1991

- Parameters of 4th-order mass-spring system
  - Uniform probability density functions for
    - \(0.5 < m_1, m_2 < 1.5\)
    - \(0.5 < k < 2\)
- Probability of Instability, \(P_i\)
  - \(m_i = 1\) (unstable) or 0 (stable)
- Probability of Settling Time Exceedance, \(P_{ts}\)
  - \(m_{ts} = 1\) (exceeded) or 0 (not exceeded)
- Probability of Control Limit Exceedance, \(P_u\)
  - \(m_u = 1\) (exceeded) or 0 (not exceeded)
- Design Cost Function
  - 10 controllers designed for the competition

\[
J = aP_i^2 + bP_{ts}^2 + cP_u^2
\]

Stochastic LQG Design for
Benchmark Control Problem

- SISO Linear-Quadratic-Gaussian Regulators
  (Marrison)
  - Implicit model following with control-rate weighting and scalar output (5th order)
  - Kalman filter with single measurement (4th order)
- Design parameters
  - Control cost function weights
  - Springs and masses in ideal model
  - Estimator weights
- Search
  - Multivariate line search
  - Genetic algorithm
Comparison of Design Costs for Benchmark Control Problem

\[ J = P_i^2 + 0.01P_i^2 + 0.01P_w^2 \]

Cost Emphasizes Instability

\[ J = 0.01P_i^2 + P_i^2 + P_w^2 \]

Cost Emphasizes Excess Control

\[ J = 0.01P_i^2 + P_i^2 + 0.01P_w^2 \]

Cost Emphasizes Settling-Time Exceedance

Stochastic LQG controller more robust in 39 of 40 benchmark controller comparisons

Estimation of Minimum Design Cost Using Jackknife/Bootstrapping Evaluation

\[ \Pr(J < J_0) \]

Corresponding Weibull Distribution

Genetic Algorithm Results

Marrison, Stengel, 1995
Neighboring-Optimal Control with Uncertain Disturbance, Measurement, and Initial Condition

Immune Response Example

- **Optimal open-loop drug therapy (control)**
  - Assumptions
    - Initial condition known without error
    - No disturbance

- **Optimal closed-loop therapy**
  - Assumptions
    - Small error in initial condition
    - Small disturbance
    - Perfect measurement of state

- **Stochastic optimal closed-loop therapy**
  - Assumptions
    - Small error in initial condition
    - Small disturbance
    - Imperfect measurement
    - Certainty-equivalence applies to perturbation control
Immune Response Example with Optimal Feedback Control

Open-Loop Optimal Control for Lethal Initial Condition

- Pathogen Kill,
- Antibody Response,
- Phagocyte Enhance.

Immune Response with Full-State Stochastic Optimal Feedback Control
(Random Disturbance and Measurement Error Not Simulated)

- Low-Bandwidth Estimator (|W| < |N|)
- High-Bandwidth Estimator (|W| > |N|)

- Initial control too sluggish to prevent divergence
- Quick initial control prevents divergence

Ghigliazza, Kulkarni, Stengel, 2002

Ghigliazza, Stengel, 2004
Stochastic-Optimal Control \((u_1)\) with Two Measurements \((x_1, x_3)\)

\[
W = I_4 \\
N = I_2 / 20
\]

Immune Response to Random Disturbance with Two-Measurement Stochastic Neighboring-Optimal Control

- Disturbance due to
  - Re-infection
  - Sequestered “pockets” of pathogen
- Noisy measurements
- Closed-loop therapy is robust
- ... but not robust enough:
  - Organ death occurs in one case
- Probability of satisfactory therapy can be maximized by stochastic redesign of controller