Flight Envelope Determined by Available Thrust

- **Flight Envelope**: Encompasses all altitudes and airspeeds at which an aircraft can fly
  - in steady, level flight
  - at fixed weight

Excess thrust provides the ability to accelerate or climb
Energy Height and Specific Excess Power

Energy Height

• Specific Energy
  • = (Potential + Kinetic Energy) per Unit Weight
  • = Energy Height

\[ \frac{\text{Total Energy}}{\text{Unit Weight}} \equiv \text{Specific Energy} = \frac{mgh + \frac{1}{2}mV^2}{mg} = \frac{h + \frac{V^2}{2g}}{2g} \]

\( \equiv \text{Energy Height}, E_h, \text{ ft or m} \)

Could trade altitude for airspeed with no change in energy height if thrust and drag were zero
Specific Excess Power

- **Specific Power**

\[
\frac{dE_h}{dt} = \frac{d}{dt} \left( h + \frac{V^2}{2g} \right) = \frac{dh}{dt} + \left( \frac{V}{g} \right) \frac{dV}{dt}
\]

\[
V = \frac{(T - D)}{m - g \sin \gamma} \\
\dot{h} = V \sin \gamma
\]

\[
\frac{dE_h}{dt} = V \sin \gamma + \left( \frac{V}{g} \right) \left[ \frac{(T - D)}{m} - g \sin \gamma \right]
\]

\[
\frac{dE_h}{dt} = V \left( \frac{T - D}{W} \right) = V \left( C_T - C_D \right) \frac{1}{2} \rho(h) V^2 S
\]

\[
\frac{dE_h}{dt} = \text{Specific Excess Power (SEP)} = \frac{\text{Excess Power}}{\text{Unit Weight}} = \frac{\left( P_{\text{thrust}} - P_{\text{drag}} \right)}{W}
\]

Contours of Constant Specific Excess Power

- Specific Excess Power is a function of altitude and airspeed
- **SEP** is maximized at each altitude, \( h \), when

\[
\frac{d[SEP(h)]}{dV} = 0
\]
Subsonic Energy Climb

Objective: Minimize time or fuel to climb to desired altitude and airspeed

Approximate Optimal Trajectory produced by a Switching Curve

Angle of Attack Control

“Point-Mass” Model

\[
\dot{V} = \left[ \frac{T_{\text{max}}}{m} - \left( C_{D_o} + \epsilon \left[ C_{L_o} \alpha \right]^2 \right) \frac{1}{2} \rho V^2 S \right] / \left( m - g \sin \gamma \right)
\]

\[
\dot{\gamma} = \frac{1}{V} \left[ \left( C_{L_o} \alpha \frac{1}{2} \rho V^2 S \right) / \left( m - g \cos \gamma \right) \right]
\]

\[
\dot{h} = V \sin \gamma
\]

\[
\dot{r} = V \cos \gamma
\]

\[
\dot{m}_{\text{fuel}} = - (SFC) (T_{\text{max}})
\]
Minimum Time-to-Climb Problem

Initial and final conditions

\[
V_o = 100 \text{ m/s}; \quad \gamma_o = 0 \text{ rad}; \quad h_o = 0 \text{ m (sea level)}; \quad r_o = 0 \text{ m}
\]

\[
V_f = 200 \text{ m/s}; \quad \gamma_f = \text{open}; \quad h_f = 10,000 \text{ m}; \quad r_f = \text{open}
\]

- End time, \(t_f\), is open, thrust takes maximum value, and control variable is angle of attack, \(\alpha(t)\).
- Fuel expended, but simulated vehicle mass held constant.

1,000-sec Trajectory, Simple Angle of Attack Control

- No optimization, open-loop control of point-mass model
- Interchange of kinetic and potential energy
- Lightly damped long-period oscillation
Altitude vs. Velocity,
*Simple Angle of Attack Control,\*  
\( t_f = 1,000 \text{ sec} \)

- Lightly damped, long-period oscillation
- Kinetic/potential energy interchange

Alternative: Pitch Angle Control

\[ \alpha = \theta - \gamma \]
\[ \alpha = \text{Angle of Attack, rad} \]
\[ \theta = \text{Pitch Angle, rad} \]
\[ \gamma = \text{Flight Path Angle, rad} \]

\[ \dot{\gamma} = \frac{1}{V} \left[ \left( C_{La} \left( \theta - \gamma \right) \frac{1}{2} \rho V^2 S \right) \right] / m - g \cos \gamma \]

Controlling pitch angle introduces **flight path angle damping**

(see Supplemental Material for details)

\[ \dot{\gamma} = \frac{1}{V} \left[ \left( C_{La} \frac{1}{2} \rho V^2 S \theta - C_{La} \frac{1}{2} \rho V^2 S \gamma \right) \right] / m - g \cos \gamma \]
Angle of Attack or Pitch Angle Control?

\[ \theta = \alpha + \gamma \]

1,000-sec Trajectory, Simple Pitch Angle Control

Note difference in pitch-angle and angle-of-attack profiles

- No optimization, open-loop control of point-mass model
- Inherent damping
- Long-period oscillation does not occur
Altitude vs. Velocity

*Simple Pitch Angle Control, \( t_f = 1,000 \text{ sec} \)

- Increased damping eliminates oscillation
- Pitch angle chosen to produce kinetic energy increase followed by potential energy increase

Minimum Time-to-Climb Optimization Problem
Minimum-Time Cost Function

\[
J(t_f) = \frac{1}{2} \left\{ \begin{array}{l}
\mathbf{x}(t_f) - \mathbf{x}_{des}(t_f) \\
\mathbf{x}(t_f) - \mathbf{x}_{des}(t_f)
\end{array} \right\}^T P_f \left\{ \begin{array}{l}
\mathbf{x}(t_f) - \mathbf{x}_{des}(t_f) \\
\mathbf{x}(t_f) - \mathbf{x}_{des}(t_f)
\end{array} \right\} + \frac{1}{2} \int_{t_0}^{t_f} \left\{ 1 + \mathbf{x}^T(t) Q_x(t) + \mathbf{u}^T(t) R_u(t) \right\} dt
\]

Terminal cost provides trajectory objective

- Integrand
  - “1” is the integrand for minimizing time
  - Small quadratic term
    - provides non-singular trajectory control with ad hoc damping and regulation [good], but
    - penalizes non-zero values of state and control [not so good]

Minimum-Time Cost Function with Augmented Trajectory Damping

\[
J(t_f) = \frac{1}{2} \left\{ \begin{array}{l}
\mathbf{x}(t_f) - \mathbf{x}_{des}(t_f) \\
\mathbf{x}(t_f) - \mathbf{x}_{des}(t_f)
\end{array} \right\}^T P_f \left\{ \begin{array}{l}
\mathbf{x}(t_f) - \mathbf{x}_{des}(t_f) \\
\mathbf{x}(t_f) - \mathbf{x}_{des}(t_f)
\end{array} \right\} + \frac{1}{2} \int_{t_0}^{t_f} \left\{ 1 + \mathbf{x}^T(t) Q_x(t) + \mathbf{x}^T(t) Q_x \mathbf{x}(t) + \mathbf{u}^T(t) R_u(t) \right\} dt
\]

\[
\dot{\mathbf{x}}(t) = \mathbf{f}[\mathbf{x}(t), \mathbf{u}(t)],
\]

- State rate weighting
  - is unbiased by non-zero state or control
  - provides damping

\[
J(t_f) = \frac{1}{2} \left\{ \begin{array}{l}
\mathbf{x}(t_f) - \mathbf{x}_{des}(t_f) \\
\mathbf{x}(t_f) - \mathbf{x}_{des}(t_f)
\end{array} \right\}^T P_f \left\{ \begin{array}{l}
\mathbf{x}(t_f) - \mathbf{x}_{des}(t_f) \\
\mathbf{x}(t_f) - \mathbf{x}_{des}(t_f)
\end{array} \right\} + \frac{1}{2} \int_{t_0}^{t_f} \left\{ 1 + \mathbf{x}^T(t) Q_x(t) + \mathbf{f}^T[\mathbf{x}(t), \mathbf{u}(t)] Q_x \mathbf{f}[\mathbf{x}(t), \mathbf{u}(t)] + \mathbf{u}^T(t) R_u(t) \right\} dt
\]
Weights Used in Optimization

\[ P = \begin{bmatrix}
100 & 0 & 0 & 0 \\
0 & 20 & 0 & 0 \\
0 & 0 & 0.4 & 0 \\
0 & 0 & 0 & 0 \\
\end{bmatrix} \]

\[ Q = \begin{bmatrix}
10^{-2} & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\end{bmatrix} \]

\[ Q_x = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\end{bmatrix} \]

\[ R = 1 \]

\[ x_1 = V: \text{Velocity, m/s} \]
\[ x_2 = \gamma: \text{Flight path angle, rad} \]
\[ x_3 = h: \text{Height, m} \]
\[ x_4 = r: \text{Range, m} \]
\[ u = \theta: \text{Pitch angle, rad} \]

Fixed End-Time Cost History,
\[ t_f = 575 \text{ sec, 36 Iterations} \]

- Ad hoc variation of final time (TBD)
- Adaptive steepest-descent optimization algorithm
- Optimization algorithm is still minimizing at iteration cutoff
Why is $t_f = 575$ sec Approximately the Minimum-Time Solution?

- $t_f = 600$ sec $J^* \approx 750$
- $t_f = 550$ sec $J^* \approx 3480$
- Desired final state not achievable with shorter time

~Minimum-Time Altitude vs. Velocity,

$t_f = 575$ sec, 36 Iterations

Kinetic energy increase
Trade for potential energy increase
Velocity Increase and Shallow Dive to satisfy terminal condition

Recall Energy-Height Approximation
Power loss with altitude greater in the simulation than in E-H approximation
Minimum-Time Trajectory, $t_f = 575$ sec, 36 Iterations

Minimum Fuel-to-Climb Optimization Problem
Minimum-Fuel Cost Function with Augmented Trajectory Damping

\[
J(t_f) = \frac{1}{2} \left[ \left( x(t_f) - x_{des}(t_f) \right)^T P_f \left[ x(t_f) - x_{des}(t_f) \right] \right] \\
+ \frac{1}{2} \int_{t_0}^{t_f} \left\{ \dot{m} + \left[ x^T(t)Qx(t) + \dot{x}^T(t)Q\dot{x}(t) + u^T(t)Ru(t) \right] \right\} dt
\]

- Same cost-function weights
- Different Integrand
  - Fuel-flow rate in the integrand for minimizing total fuel use

Minimum-Fuel Cost History, \( t_f = 575 \text{ sec}, 36 \text{ Iterations} \)

Adaptive optimization algorithm takes two bad iterations... but regains convergence, with small oscillation in terminal cost
Minimum-Fuel Altitude vs. Velocity, 
$t_f = 575$ sec, 36 Iterations

~Minimum-Fuel Trajectory, 
$t_f = 575$ sec, 36 Iterations

Virtually identical to minimum-time solution 
- Minimum time: 849 kg 
- Minimum fuel: 831 kg

... but less fuel is used 
- Minimum time: 849 kg 
- Minimum fuel: 831 kg
Minimum-Fuel Adjoint Vector History, $\lambda(t)$, $t_f = 575$ sec, 36 Iterations

Cost sensitivity to state perturbations over time

$\lambda(t_f) = \left[ \frac{\partial \phi}{\partial x} (t_f) \right]^T = \Delta x^T (t_f) P_f$

Next Time:

Neighboring-Optimal Control via Linear-Quadratic Feedback

Reading

OCE: Section 3.7
Modal Properties of the System
Linearized Equations for Velocity and Flight Path Angle Perturbations, Using Angle of Attack as the Control

\[
\begin{bmatrix}
\Delta V \\
\Delta \gamma
\end{bmatrix} = \begin{bmatrix}
(T_v - D_v) & -g \\
L_v/V_o & 0
\end{bmatrix} \begin{bmatrix}
\Delta V \\
\Delta \gamma
\end{bmatrix} + \begin{bmatrix}
0 \\
L_v/V_o
\end{bmatrix} \Delta \alpha
\]

State:  \( \Delta x = \begin{bmatrix} \Delta V \\ \Delta \gamma \end{bmatrix} \)

Control:  \( \Delta u = \Delta \alpha \)

\( T_v = \frac{\partial (\text{Thrust/m})}{\partial V} = \) Sensitivity of acceleration-due-to-thrust to velocity variation

\( = 0 \) for this problem because [\text{thrust = maximum thrust}]\]

\( D_v = \frac{\partial (\text{Drag/m})}{\partial V} = \) Sensitivity of acceleration-due-to-drag to velocity variation  > 0

\( L_v = \frac{\partial (\text{Lift/m})}{\partial V} = \) Sensitivity of acceleration-due-to-lift to velocity variation  > 0

\( L_\alpha = \frac{\partial (\text{Lift/m})}{\partial \alpha} = \) Sensitivity of acceleration-due-to-lift to angle-of-attack variation  > 0

Characteristic Equation and Stability, Using Angle of Attack as the Control Variable

\[
[sI - F] = s \begin{bmatrix}
-D_v & -g \\
L_v/V_o & 0
\end{bmatrix} = \begin{bmatrix}
(s + D_v) & g \\
L_v/V_o & s
\end{bmatrix}
\]

\[
s(s + D_v) + g \frac{L_v}{V_o} = 0
\]

\[
s^2 + D_v s + g \frac{L_v}{V_o} = 0
\]

\[
s^2 + 2\omega_a s + \omega_a^2 = 0
\]

\[
0 = s^2 + 2\zeta\omega_a s + \omega_a^2
\]

\[
\omega_a = \sqrt{g \frac{L_v}{V_o}} = \sqrt{\frac{2}{V_o(m/s)}} = \frac{13.9}{V_o(m/s)} ; \text{ Period } = 0.453 V_o, \text{ sec}
\]

\[
\zeta = \frac{D_v}{2 \sqrt{g \frac{L_v}{V_o}}} = \sqrt{\frac{1}{2} \left( \frac{C_D}{C_L} \right)}
\]

"Total Damping" = \( 2\zeta\omega_a = D_v \)
Linearized Equations for Velocity and Flight Path Angle Perturbations, Using Pitch Angle as the Control

\[
\begin{bmatrix}
\Delta \dot{V} \\
\Delta \dot{\gamma}
\end{bmatrix} =
\begin{bmatrix}
-D_v & -g \\
L_v/V_o & 0
\end{bmatrix}
\begin{bmatrix}
\Delta V \\
\Delta \gamma
\end{bmatrix} + \begin{bmatrix}
0 \\
L_o/V_o
\end{bmatrix}\Delta \alpha
\]

State: \( \Delta x = \begin{bmatrix} \Delta V \\ \Delta \gamma \end{bmatrix} \)

Control: \( \Delta u = \Delta \theta = \Delta \alpha + \Delta \gamma \)

Replace angle of attack by pitch angle for control

\[
\begin{bmatrix}
\Delta \dot{V} \\
\Delta \dot{\gamma}
\end{bmatrix} =
\begin{bmatrix}
-D_v & -g \\
L_v/V_o & 0
\end{bmatrix}
\begin{bmatrix}
\Delta V \\
\Delta \gamma
\end{bmatrix} + \begin{bmatrix}
0 \\
L_o/V_o
\end{bmatrix}(\Delta \theta - \Delta \gamma)
= \begin{bmatrix}
-D_v & -g \\
L_v/V_o & -L_o/V_o
\end{bmatrix}
\begin{bmatrix}
\Delta V \\
\Delta \gamma
\end{bmatrix} + \begin{bmatrix}
0 \\
L_o/V_o
\end{bmatrix}\Delta \theta
\]

Characteristic Equation and Stability, Using Pitch Angle as the Control Variable

\[
[M - F] = sI - \begin{bmatrix}
-D_v & -g \\
L_v/V_o & -L_o/V_o
\end{bmatrix} = (s + D_v)\begin{bmatrix}
g \\
L_v/V_o
\end{bmatrix} = s^2 + L_o/V_o \left( s + D_v \right) + gL_v/V_o
= s^2 + L_o/V_o \left( s + D_v \right) + gL_v/V_o
= s^2 + 2\zeta\omega_n s + \omega_n^2 = 0
\]

\[\omega_n = \sqrt{g L_v/V_o + D_v L_o/V_o}
\]

\[\zeta = \frac{\left( L_o/V_o + D_v \right)}{2\sqrt{g L_v/V_o + D_v L_o/V_o}}\]

- Natural frequency is increased
- Damping is increased

"Total Damping" = \( 2\zeta\omega_n = \left( L_o/V_o + D_v \right) \)
Definitions and Numerical Values for Variables and Constants

$V = \text{Velocity, m/s}$  
$\gamma = \text{Flight path angle, rad}$  
$h = \text{Altitude (or height), m}$  
$r = \text{Range, m}$  
$m = \text{Mass, kg}$  
$\alpha = \text{Angle of attack, rad}$  
$\alpha_{\text{max}} = 10^\circ$  
$\rho = \text{Air density} = \rho_{\text{sl}} e^{-\beta h} = 1.225 e^{-0.53h} \text{kg/m}^3$  
$g = \text{Gravitational acceleration} = 9.801 \text{m/s}^2$

$T = T_{\text{sl}} \left( \frac{\rho}{\rho_{\text{sl}}} \right) \delta T = T_{\text{sl}} \left( e^{-\beta h} \right) \delta T = \text{Thrust, N}$

$\delta T = \text{Throttle setting, } \%c, = 100\% \text{ for minimum-time/fuel problem}$

$SFC = \text{Specific Fuel Consumption} = 10 \text{ g/kN-s}$

$C_L = C_{\alpha} \alpha = \text{Lift coefficient} = 5.7\alpha$

$C_D = \left( \frac{C_D + \varepsilon C_i^2}{\sqrt{1-(V/V_{\text{sound}})^2}} \right) = \text{Drag coefficient}$

$= \left( 0.025 + 0.072C_i^2 \right) \sqrt{1-(V/V_{\text{sound}})^2}$

$S = \text{Reference area} = 21.5 \text{ m}^2$

$m_E = \text{Vehicle mass} = 4,550 \text{ kg} \sim \text{constant}$

Angle of attack has linear effect on lift and quadratic effect on drag

Supersonic Energy Climb

Objective: Minimize time or fuel to climb to desired altitude and airspeed
Energy State Profile,
\( t_f = 570 \) sec

Energy State:
\[
E = \frac{1}{mg} \left( \frac{mV^2}{2} + mgh \right) = \frac{V^2}{2g} + h, \quad \text{meters}
\]

- Optimized Energy State is monotonic and always increasing
- Rate of change decreases with altitude

Specific Excess Power:
\[
SEP = \frac{V}{mg} (\text{Thrust} - \text{Drag})
\]