Time-Invariant Linear Quadratic Regulators
Robert Stengel
Optimal Control and Estimation MAE 546
Princeton University, 2017

- Asymptotic approach from time-varying to constant gains
- Elimination of cross weighting in cost function
- Controllability and observability of an LTI system
- Requirements for closed-loop stability
- Algebraic Riccati equation
- Equilibrium response to commands

Continuous-Time, Linear, Time-Invariant System Model

\[ \Delta x(t) = F \Delta x(t) + G \Delta u(t) + L \Delta w(t), \]
\[ \Delta x(t_0) \text{ given} \]
\[ \Delta y(t) = H_x \Delta x(t) + H_u \Delta u(t) + H_w \Delta w(t) \]

Comment: \( \Delta (\cdot) \) notation distinguishes linear-system variables from nonlinear-system variables
Linear-Quadratic Regulator:
Finite Final Time

\[ \Delta \dot{x}(t) = F \Delta x(t) + G \Delta u(t) \]

\[ \Delta u(t) = -R^{-1} \left[ M^T + G^T P(t) \right] \Delta x(t) = -C(t) \Delta x(t) \]

\[ \Delta^2 J = \frac{1}{2} \left[ \Delta x^T(t_f) P(t_f) \Delta x(t_f) \right] \]

\[ P(t) = -\left[ F - GR^{-1} M^T \right]^T P(t) - P(t) \left[ F - GR^{-1} M^T \right] + P(t) GR^{-1} G^T P(t) + \left[ MR^{-1} M^T - Q \right] \]

\[ P(t_f) = \bar{P}_f \]

Transformation of Variables to Eliminate Cost Function Cross Weighting

Original LTI minimization problem

\[ \min_{\Delta u_i} J_1 = \frac{1}{2} \int_0^{t_f} \left[ \Delta x_i^T(t) Q_1 \Delta x_i(t) + \Delta x_i^T(t) M_i \Delta u_i(t) + \Delta u_i(t) R_i \Delta u_i(t) \right] dt \]

subject to \[ \Delta \dot{x}_i(t) = F_i \Delta x_i(t) + G_i \Delta u_i(t) \]

Can we find a transformation such that

\[ \min_{\Delta u_2} J_2 = \frac{1}{2} \int_0^{t_f} \left[ \Delta x_2^T(t) Q_2 \Delta x_2(t) + \Delta u_2^T(t) R_2 \Delta u_2(t) \right] dt = \min_{\Delta u_1} J_1 \]

subject to \[ \Delta \dot{x}_2(t) = F_2 \Delta x_2(t) + G_2 \Delta u_2(t) \]
Artful Manipulation

Rewrite integrand of $J_1$ to eliminate cross weighting of state and control

\[
\begin{align*}
\Delta x_1^T(t)Q_1\Delta x_1(t) + 2\Delta x_1^T(t)M_1\Delta u_1(t) + \Delta u_1(t)R_1\Delta u_1(t) \\
= \Delta x_1^T(t)\left(Q_1 - M_1R_1^{-1}M_1^T\right)\Delta x_1(t) \\
+ \left[\Delta u_1(t) + R_1^{-1}M_1^T\Delta x_1(t)\right]^T R_1\left[\Delta u_1(t) + R_1^{-1}M_1^T\Delta x_1(t)\right]
\end{align*}
\]

\[
\Delta x_1^T(t)Q_2\Delta x_1(t) + \Delta u_2^T(t)R_1\Delta u_2(t)
\]

The transformation produces the following equivalences

\[
\Delta x_2(t) = \Delta x_1(t) \\
\Delta u_2(t) = \Delta u_1(t) + R_1^{-1}M_1^T\Delta x_1(t) \\
Q_2 = Q_1 - M_1R_1^{-1}M_1^T \\
R_2 = R_1
\]

(Q,R) and (Q,M,R) LQ Problems are Equivalent

\[
\Delta x_2(t) = \Delta x_1(t) \Rightarrow \\
\Delta \dot{x}_2(t) = \Delta \dot{x}_1(t) \\
\Delta u_2(t) = \Delta u_1(t) + R_1^{-1}M_1^T\Delta x_1(t) \\
Q_2 = Q_1 - M_1R_1^{-1}M_1^T \\
R_2 = R_1
\]

\[
\begin{align*}
\Delta \dot{x}_2(t) &= F_2\Delta x_2(t) + G_2\Delta u_2(t) \\
\Delta \dot{x}_2(t) &= F_2\Delta x_1(t) + G_2\left[\Delta u_1(t) + R_1^{-1}M_1^T\Delta x_1(t)\right] \\
&= \left(F_2 + R_1^{-1}M_1^T\right)\Delta x_1(t) + G_2\Delta u_1(t) \\
&= \Delta \dot{x}_1(t) = F_1\Delta x_1(t) + G_1\Delta u_1(t)
\end{align*}
\]

\[
\begin{align*}
G_2 &= G_1 \\
F_2 &= F_1 - G_2R_1^{-1}M_1^T \\
&= F_1 - G_1R_1^{-1}M_1^T
\end{align*}
\]

- Therefore, the 2 forms are equivalent
- Whatever we prove for a (Q,R) cost function pertains to a (Q,M,R) cost function
**Recall:** LQ Optimal Control of an **Unstable** First-Order System

\[ f = 1; \quad g = 1 \]

\[ \Delta \dot{x} = \Delta x + \Delta u; \quad x(0) = 1 \]

\[ \dot{p}(t) = -1 - 2p(t) + p^2(t) \]

\[ p(t_f) = 1 \]

Control gain = \( p(t) \)

\[ \Delta u = -p(t) \Delta x \]

\[ \Delta \dot{x} = [1 - p(t)] \Delta x \]

---

**Riccati Solution and Control Gain**

for Open-Loop **Stable** and **Unstable** 1\(^{st}\)-Order Systems

\[ P(t_f) = 0 \]

Variations in control gains are significant only in the
last 10-20% of the illustrated time interval
As time interval increases, percentage decreases
P(0) Approaches Steady State as $t_f \to \infty$

With $M = 0$, $P(t_f) = 0$

\[
P(0) = -\int_{t_f}^{0} \left\{ -Q - F^T P(t) - P(t) F + P(t) GR^{-1} G^T P(t) \right\} dt
\]

from $t_f$ to $0$

Progression of initial Riccati matrix is monotonic with increasing final time

Rate of change approaches zero with increasing final time

For $t_{f_2} > t_{f_1}$,

\[
J^*(t_{f_2}) \geq J^*(t_{f_1}), \quad \therefore P_2(0) \geq P_1(0)
\]

(see eq. 5.4-9 to 5.4-11, OCE)

Algebraic Riccati Equation and Constant Control Gain Matrix

Steady-state Riccati solution ($M = 0$)

\[
-Q - F^T P(0) - P(0) F + P(0) GR^{-1} G^T P(0) = 0
\]

\[
-Q - F^T P_{SS} - P_{SS} F + P_{SS} GR^{-1} G^T P_{SS} = 0
\]

Steady-state control gain matrix

\[
C_{ss} = R^{-1} G^T P(0 | t_f \to \infty) = R^{-1} G^T P_{ss}
\]
Controllability of a LTI System

**Controllability**: All elements of the state can be brought from arbitrary initial conditions to zero in finite time

\[ \Delta x(t) = F \Delta x(t) + G \Delta u(t) \]
\[ \Delta x(0) = \Delta x_0; \quad \Delta x(t_{\text{finite}}) = 0 \]

System is Completely Controllable if

Controllability Matrix =

\[
\begin{bmatrix}
G & FG & \cdots & F^{n-1}G
\end{bmatrix}
\]

has Rank \( n \)

Controllability Examples

For non-zero coefficients

For \( F = \begin{bmatrix} 0 & 0 & 0 \\ -\omega_n^2 & -2\zeta\omega_n & 0 \\ 0 & 0 & \omega_n^2 \end{bmatrix} \)

\( G = \begin{bmatrix} 0 & 0 & 0 \\ \omega_n^2 & 0 & -2\zeta\omega_n^2 \end{bmatrix} \)

\[
\begin{bmatrix}
G & FG \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ \omega_n^2 & 0 & -2\zeta\omega_n^2 \end{bmatrix} \Rightarrow \text{Rank} = 2
\]

For \( F = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ b & 0 & 0 \end{bmatrix} \)

\( G = \begin{bmatrix} 0 & 0 \end{bmatrix} \)

\[
\begin{bmatrix}
G & FG \end{bmatrix} = \begin{bmatrix} 0 & 0 \end{bmatrix} \Rightarrow \text{Rank} = 1
\]

For \( F = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & b \end{bmatrix} \)

\( G = \begin{bmatrix} 0 & 0 \end{bmatrix} \)

\[
\begin{bmatrix}
G & FG \end{bmatrix} = \begin{bmatrix} 0 & 0 \end{bmatrix} \Rightarrow \text{Rank} = 2
\]

For \( F = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & b \end{bmatrix} \)

\( G = \begin{bmatrix} 0 & 0 \end{bmatrix} \)

\[
\begin{bmatrix}
G & FG \end{bmatrix} = \begin{bmatrix} 0 & 0 \end{bmatrix} \Rightarrow \text{Rank} = 2
\]
Requirements for Guaranteed Closed-Loop Stability

Optimal Cost with Feedback Control

With terminal cost = 0

\[
J^*(t_f) = \frac{1}{2} \int_{0}^{t_f} \left[ \Delta x^* (t) Q \Delta x^* (t) + \Delta u^* (t) R \Delta u^* (t) \right] dt
\]

Substitute optimal control law in cost function

\[
= \frac{1}{2} \int_{0}^{t_f} \left[ \Delta x^* (t) Q \Delta x^* (t) + \Delta x^* (t) C^T (t) R C (t) \Delta x^* (t) \right] dt
\]
Optimal Cost with LQ Feedback Control

Consolidate terms

\[
J^*(t_f) = \frac{1}{2} \int_0^{t_f} [\Delta x^T(\tau) [Q + C^T(\tau)RC(\tau)] \Delta x(\tau)] d\tau
\]

From eq. 5.4-9, OCE, optimal cost depends only on the initial state and

\[
J(t_f) = \frac{1}{2} \Delta x^T(0)P(0)\Delta x(0)
\]

Optimal Quadratic Cost Function is Bounded with LQ Feedback Control

As final time goes to infinity

\[
J^*(\infty) = \lim_{{t_f \to \infty}} \frac{1}{2} \int_0^{t_f} [\Delta x^T(\tau) [Q + C^T(\tau)RC(\tau)] \Delta x(\tau)] d\tau
\]

\[
\triangleq \frac{1}{2} \int_0^{\infty} [\Delta x^T(\tau) [Q + C^TRC] \Delta x(\tau)] d\tau = \frac{1}{2} \Delta x^T(0)P\Delta x(0)
\]

\[
J \text{ is bounded and positive provided that } \begin{cases} Q > 0 \\ R > 0 \end{cases}
\]

Because \( J \) is bounded, optimal gain, \( C \), is a stabilizing gain matrix
Requirements for Guaranteeing Stability of the LQ Regulator

$$\Delta \dot{x}(t) = F \Delta x(t) + G \Delta u(t) = [F - GC] \Delta x(t)$$

Closed-loop system is stable whether or not open-loop system is stable if ...

\[
\begin{align*}
Q &> 0 \\
R &> 0
\end{align*}
\]

... and \((F,G)\) is a controllable pair

\[
\text{Rank} \begin{bmatrix} G & FG & \cdots & F^{n-1}G \end{bmatrix} = n
\]

Lyapunov Stability of the LQ Regulator

\[
\Delta \dot{x}(t) = [F - GC] \Delta x(t) = [F - GR^{-1}G^T P] \Delta x(t)
\]

Lyapunov function

\[
V[\Delta x(t)] = \Delta x^T(t) P \Delta x(t) \geq 0
\]

Rate of change of Lyapunov function

\[
\dot{V} = \Delta x^T(t) P \Delta \dot{x}(t) + \Delta \dot{x}^T(t) P \Delta x(t) \\
= \Delta x^T(t) \left\{ P [F - GR^{-1}G^T P] + [F - GR^{-1}G^T P]^T P \right\} \Delta x(t)
\]
Lyapunov Stability of the LQ Regulator

Algebraic Riccati equation

\[-Q - F^T P - PF + PGR^{-1}G^T P = 0\]

Substituting in rate equation

\[\dot{V} = \Delta x^T(t) \left\{ P \left[ F - GR^{-1}G^T P \right] + \left[ F - GR^{-1}G^T P \right]^T \right\} \Delta x(t)\]

\[= -\Delta x^T(t) \left\{ Q + PGR^{-1}G^T P \right\} \Delta x(t) \leq 0\]

Defining matrix is positive definite
Therefore, closed-loop system is stable

Less Restrictive Stability Requirements

\(Q\) may be positive semi-definite if \((F,D)\) is an observable pair, where

\[Q \triangleq D^T D, \text{ where } D \text{ may not be } (n \times n)\]

Observability requirement

\[
\text{Rank} \left[ \begin{array}{cccc}
D^T & F^T D^T & \cdots & \left(F^T\right)^{n-1} D^T
\end{array} \right] = n
\]
Observability Example

\[
\begin{bmatrix}
\Delta \dot{x}_1(t) \\
\Delta \dot{x}_2(t)
\end{bmatrix} =
\begin{bmatrix}
0 & 1 \\
-\omega_n^2 & -2\zeta \omega_n
\end{bmatrix}
\begin{bmatrix}
\Delta x_1(t) \\
\Delta x_2(t)
\end{bmatrix} = F \Delta x(t)
\]

\[
\Delta y(t) =
\begin{bmatrix}
0 & 1
\end{bmatrix}
\begin{bmatrix}
\Delta x_1(t) \\
\Delta x_2(t)
\end{bmatrix} = H \Delta x(t)
\]

For non-zero coefficients, with H = D

\[
\begin{bmatrix}
H^T & F^T H^T
\end{bmatrix} =
\begin{bmatrix}
0 & -\omega_n^2 \\
1 & -2\zeta \omega_n
\end{bmatrix} \Rightarrow \text{Rank} = 2
\]

Even Less Restrictive Stability Requirements

- If F contains stable modes, closed-loop stability is guaranteed if
  - \((F,G)\) is a stabilizable pair
  - \((F,D)\) is a detectable pair
Stability Requirements with Cross Weighting, $M$, in Cost Function

- If $F$ contains stable modes, closed-loop stability is guaranteed if
  - $[(F – GR^{-1}M^T),G]$ is a **stabilizable** pair
  - $[(F – GR^{-1}M^T),D]$ is a **detectable** pair
  - $(Q – GR^{-1}M^T) \geq 0$
  - $R > 0$

Example: LQ Optimal Control of a First-Order LTI System

**Cost Function**

$$
\Delta^2 J = \frac{1}{2} \dot{\Delta x}^2(t_f) + \lim_{t_f \to \infty} \frac{1}{2} \int_{t_i}^{t_f} \left( q \Delta x^2 + r \Delta u^2 \right) dt
$$

**Open-Loop System**

$$
\dot{\Delta x} = f \Delta x + g \Delta u
$$

**Control Law**

$$
\Delta u = -\frac{gp}{r} \Delta x = -c \Delta x
$$

**Algebraic Riccati Equation**

$$
-q - 2fp + \frac{g^2p^2}{r} = 0
$$

$$
p^2 - 2\frac{fr}{g^2}p - \frac{qr}{g^2} = 0
$$

**Choose positive solution of**

$$
p = \frac{fr}{g^2} \pm \sqrt{\left( \frac{fr}{g^2} \right)^2 + \frac{qr}{g^2}}
$$

$$
= \frac{fr}{g^2} \left[ 1 \pm \sqrt{1 + \frac{qg^2}{f^2r}} \right]
$$
Example: LQ Optimal Control of a First-Order LTI System

**Closed-Loop System**

\[
\Delta \dot{x} = \left( f - \frac{g^2 p}{r} \right) \Delta x = (f - c) \Delta x
\]

Stability requires that

\[
(f - c) < 0
\]

If \( f < 0 \), then the system is stable with no control \((c = 0)\)

Example: LQ Optimal Control of a First-Order LTI System

If \( f > 0 \) (unstable), and \( r > 0 \), then \( \frac{fr}{g^2} > 0 \), and

\[
p = \frac{fr}{g^2} \left[ 1 \pm \sqrt{1 + \frac{qg^2}{f^2 r}} \right]
\]

If \( q \geq 0 \), and \( g \neq 0 \), then

\[
p \xrightarrow{q \to 0} \frac{fr}{g^2} \left[ 1 + \sqrt{1} \right] = \frac{2fr}{g^2}
\]

and closed-loop system is, as \( q \to 0 \),

\[
(f - \frac{g^2}{r} p) = \left( f - \frac{g^2}{r} \frac{2fr}{g^2} \right) = (f - 2f) = -f
\]

Stable closed-loop system is "mirror image" of unstable open-loop system when \( q = 0 \)
Caveat

• If …

\[(F, G)_{\text{assumed}} \neq (F, G)_{\text{actual}}\]

• … all bets about optimal feedback control stability are off (for the moment)

• The robustness of optimal control with parameter uncertainty remains to be determined

Solution of the Algebraic Riccati Equation
Solution Methods for the Continuous-Time Algebraic Riccati Equation

\[-Q - F^T P - PF + PGR^{-1}G^T P = 0\]

1) Integrate Riccati differential equation to steady state

2) Explicit scalar equations for elements of \( P \)
   a) Difficult for \( n > 3 \)
   b) May use symbolic math (MATLAB Symbolic Math Toolbox, Mathematica, ...)

Example: Scalar Solution for the Algebraic Riccati Equation

\[-Q - F^T P - PF + PGR^{-1}G^T P = 0\]

Second-order example

\[- \begin{bmatrix} q_{11} & 0 \\ 0 & q_{22} \end{bmatrix} - \begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix}^{T} \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} - \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} \begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix} + \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} \begin{bmatrix} r_{11} & 0 \\ 0 & r_{22} \end{bmatrix}^{-1} \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix}^{T} \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} = 0\]

Solve three scalar equations for \( p_{11}, p_{12}, \) and \( p_{22} \)
More Solutions for the Algebraic Riccati Equation

\[-Q - F^T P - PF + PGR^{-1}G^T P = 0\]

- See *OCE, Section 6.1* for
  - Kalman-Englar method
  - Kleinman’s method
  - MacFarlane-Potter method
  - Laub’s method [used in MATLAB]

Equilibrium Response to a Command Input
Steady-State Response to Command Input of Open-Loop System

\[ \Delta x(t) = F \Delta x(t) + G \Delta u(t) + L \Delta w(t), \]
\[ \Delta x(t_0) \text{ given} \]
\[ \Delta y(t) = H_x \Delta x(t) + H_u \Delta u(t) + H_w \Delta w(t) \]

State equilibrium with constant inputs (\( \cdot \))\(^* \) ...

\[ 0 = F \Delta x^* + G \Delta u^* + L \Delta w^* \]
\[ \Delta x^* = -F^{-1}(G \Delta u^* + L \Delta w^*) \]

... constrained by requirement to satisfy command input

\[ \Delta y^* = H_x \Delta x^* + H_u \Delta u^* + H_w \Delta w^* \]

Steady-State Response to Commands

Equilibrium that satisfies a command input, \( \Delta y_C \)

\[ 0 = F \Delta x^* + G \Delta u^* + L \Delta w^* \]
\[ \Delta y^* = H_x \Delta x^* + H_u \Delta u^* + H_w \Delta w^* \]

Combine equations for simultaneous solution

\[
\begin{bmatrix}
0 \\
\Delta y_C
\end{bmatrix} =
\begin{bmatrix}
F & G \\
H_x & H_u
\end{bmatrix}
\begin{bmatrix}
\Delta x^* \\
\Delta u^*
\end{bmatrix} +
\begin{bmatrix}
L \\
H_w
\end{bmatrix}
\Delta w^*
\]

\((n + r) \times (n + m)\)
Equilibrium Values of State and Control to Satisfy Command Input

Equilibrium that satisfies a commanded input, $\Delta y_c$

\[
\begin{bmatrix}
\Delta x^* \\
\Delta u^*
\end{bmatrix} = \begin{bmatrix} F & G \\ H_x & H_u \end{bmatrix}^{-1} \begin{bmatrix}
-\Delta L \Delta w^* \\
\Delta y_c - H \Delta w^*
\end{bmatrix}
\]

\[\Delta \equiv A^{-1} \begin{bmatrix}
-\Delta L \Delta w^* \\
\Delta y_c - H \Delta w^*
\end{bmatrix}\]

$A$ must be square for inverse to exist

Then, number of commands = number of controls

Inverse of the Matrix

\[
\begin{bmatrix} F & G \\ H_x & H_u \end{bmatrix}^{-1} \triangleq A^{-1} = B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}
\]

\[
\begin{bmatrix}
\Delta x^* \\
\Delta u^*
\end{bmatrix} = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} \begin{bmatrix}
-\Delta L \Delta w^* \\
\Delta y_c - H \Delta w^*
\end{bmatrix}
\]

$B_{ij}$ have same dimensions as equivalent blocks of $A$

Equilibrium that satisfies a commanded input, $\Delta y_c$

\[
\Delta x^* = -B_{11} \Delta L \Delta w^* + B_{12} \left( \Delta y_c - H \Delta w^* \right)
\]

\[
\Delta u^* = -B_{21} \Delta L \Delta w^* + B_{22} \left( \Delta y_c - H \Delta w^* \right)
\]
Elements of Matrix Inverse and Solutions for Open-Loop Equilibrium

Substitution and elimination (see Supplement)

\[
\begin{bmatrix}
B_{11} & B_{12} \\
B_{21} & B_{22}
\end{bmatrix}
= 
\begin{bmatrix}
F^{-1}(-GB_{21} + I_n) & -F^{-1}GB_{22} \\
-B_{22}H_xF^{-1} & (-H_xF^{-1}G + H_u)^{-1}
\end{bmatrix}
\]

Solve for \(B_{22}\), then \(B_{12}\) and \(B_{21}\), then \(B_{11}\)

\[\Delta x^* = B_{12}\Delta y_C - (B_{11}L + B_{12}H_w)\Delta w^*\]

\[\Delta u^* = B_{22}\Delta y_C - (B_{21}L + B_{22}H_w)\Delta w^*\]

LQ Regulator with Command Input (Proportional Control Law)

\[\Delta u(t) = \Delta u_C(t) - C\Delta x(t)\]

How do we define \(\Delta u_C(t)\)?
Non-Zero Steady-State Regulation with LQ Regulator

Command input provides equivalent state and control values for the LQ regulator

\[
\Delta u(t) = \Delta u^*(t) - C \left[ \Delta x(t) - \Delta x^*(t) \right]
= B_{22} \Delta y^* - C \Delta x(t) - B_{12} \Delta y^*
= (B_{22} + CB_{12}) \Delta y^* - C \Delta x(t)
\]

Control law with command input

Disturbance affects the system, whether or not it is measured
If measured, disturbance effect of can be countered by \( C_D \) (analogous to \( C_F \))
Next Time:
Cost Functions and Controller Structures

Supplemental Material
Square-Root Solution for the Algebraic Riccati Equation

\[-Q - F^T P - PF + P G R^{-1} G^T P = 0\]

Square root of \( P \):

\[P \triangleq D D^T; \quad D \triangleq \sqrt{P}\]

Integrate \( D \) to steady state

\[\dot{D}(t) = D^T M_{LT}(t), \quad D(t_f) D^T(t_f) = P(t_f | t_f \to \infty)\]

\[M(t) \triangleq M_{LT}(t) + M_{UT}(t)\]

\[= -D^{-1}(t) F^T D(t) - D^T(t) F^T D^{-1}(t) - D^{-1}(t) Q D^T(t) + D^T(t) G R^{-1} G^T D^{-1}(t)\]

\[\Delta u(t) = -R^{-1} \left[ G^T D_{SS} D_{SS}^T \right] \Delta x(t)\]

\[= -C_{SS} \Delta x(t)\]

Matrix Inverse Identity

OCE, eq. 2.2-57 to -67

\[
\begin{bmatrix}
B_{11} & B_{12} \\
B_{21} & B_{22}
\end{bmatrix}
\begin{bmatrix}
A_{11} & A_{12} \\
A_{21} & A_{22}
\end{bmatrix}
\triangleq I_{m+n} =
\begin{bmatrix}
I_n & 0 \\
0 & I_m
\end{bmatrix}
\]

\[
\begin{bmatrix}
B_{11} & B_{12} \\
B_{21} & B_{22}
\end{bmatrix}
\begin{bmatrix}
A_{11} & A_{12} \\
A_{21} & A_{22}
\end{bmatrix} =
\begin{bmatrix}
(B_{11} A_{11} + B_{12} A_{21}) & (B_{11} A_{12} + B_{12} A_{22}) \\
(B_{21} A_{11} + B_{22} A_{21}) & (B_{21} A_{12} + B_{22} A_{22})
\end{bmatrix}
\]

\[
(B_{11} A_{11} + B_{12} A_{21}) = I_n
\]

\[
(B_{11} A_{12} + B_{12} A_{22}) = 0
\]

\[
(B_{21} A_{11} + B_{22} A_{21}) = 0
\]

\[
(B_{21} A_{12} + B_{22} A_{22}) = I_m
\]

Solve for \( B_{22} \), then \( B_{12} \) and \( B_{21} \), then \( B_{11} \)