Relationship Between Euler-Angle Rates and Body-Axis Rates

- $\dot{\psi}$ is measured in the Inertial Frame
- $\dot{\theta}$ is measured in Intermediate Frame #1
- $\dot{\phi}$ is measured in Intermediate Frame #2
- **... which is**

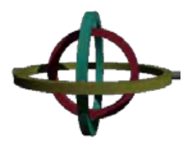
$$\begin{bmatrix} p \\ q \\ r \end{bmatrix} = \mathbf{I}_3 \begin{bmatrix} \dot{\phi} \\ 0 \\ 0 \end{bmatrix} + \mathbf{H}_2^B \begin{bmatrix} 0 \\ \dot{\theta} \\ 0 \end{bmatrix} + \mathbf{H}_2^B \mathbf{H}_1^2 \begin{bmatrix} 0 \\ 0 \\ \dot{\psi} \end{bmatrix}$$

$$\begin{bmatrix} p \\ q \\ r \end{bmatrix} = \begin{bmatrix} 1 & 0 & -\sin\theta \\ 0 & \cos\phi & \sin\phi\cos\theta \\ 0 & -\sin\phi & \cos\phi\cos\theta \end{bmatrix} \begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} = \mathbf{L}_I^B \dot{\mathbf{\Theta}}$$

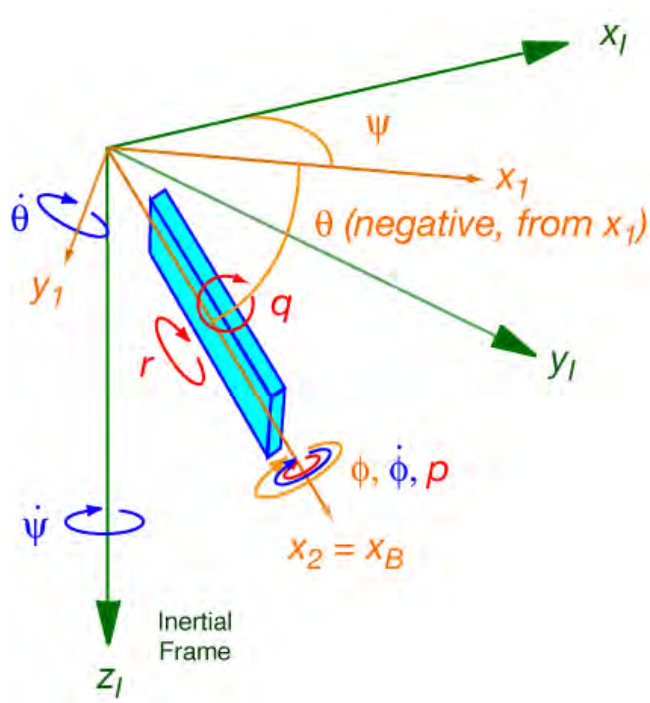
Can the inversion become singular?
What does this mean?

Inverse transformation $[(.)^{-1} \neq (.)^T]$

$$\begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} = \begin{bmatrix} 1 & \sin\phi\tan\theta & \cos\phi\tan\theta \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi\sec\theta & \cos\phi\sec\theta \end{bmatrix} \begin{bmatrix} p \\ q \\ r \end{bmatrix} = \mathbf{L}_B^I \boldsymbol{\omega}_B$$



Euler-Angle Rates and Body-Axis Rates



Avoiding the Euler Angle Singularity at $\theta = \pm 90°$

- **Alternatives to Euler angles**
  - Direction cosine (rotation) matrix
  - Quaternions

Propagation of direction cosine matrix (9 parameters)

$$\dot{\mathbf{H}}_B^I \mathbf{h}_B = \tilde{\boldsymbol{\omega}}_I \mathbf{H}_B^I \mathbf{h}_B$$

Consequently

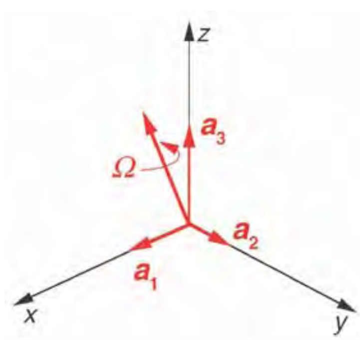
$$\dot{\mathbf{H}}_I^B(t) = -\tilde{\boldsymbol{\omega}}_B(t)\mathbf{H}_I^B(t) = -\begin{bmatrix} 0 & -r(t) & q(t) \\ r(t) & 0 & -p(t) \\ -q(t) & p(t) & 0(t) \end{bmatrix}_B \mathbf{H}_I^B(t)$$

$$\mathbf{H}_I^B(0) = \mathbf{H}_I^B(\phi_0, \theta_0, \psi_0)$$

Avoiding the Euler Angle Singularity at $\theta = \pm 90°$

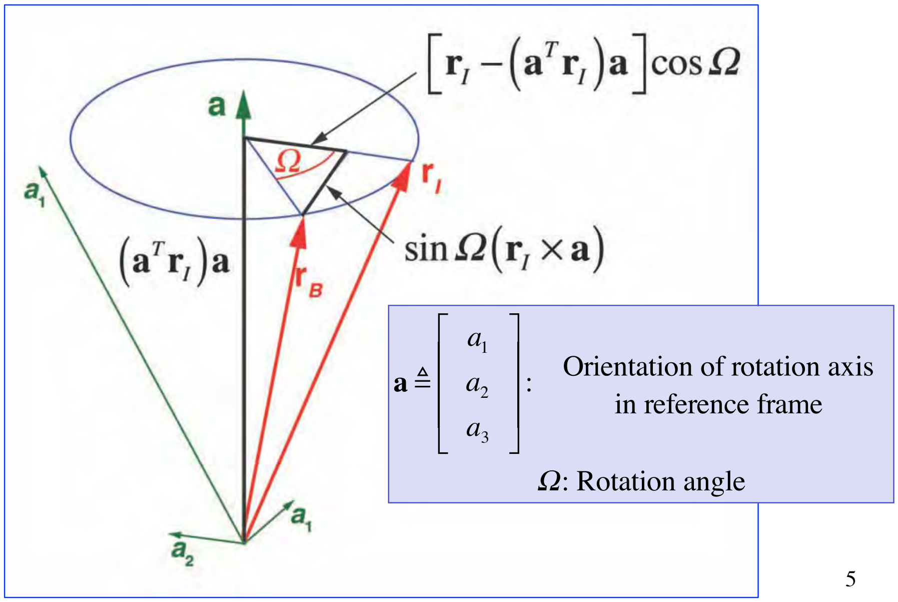
Propagation of quaternion vector: single rotation from inertial to body frame (4 parameters)

- **Rotation from one axis system, *I*, to another, *B*, represented by**
  - Orientation of axis vector about which the rotation occurs (3 parameters of a unit vector, a_1, a_2, and a_3)
  - Magnitude of the rotation angle, Ω, rad



Euler Rotation of a Vector

Rotation of a vector to an arbitrary new orientation can be expressed as a single rotation about an axis at the vector's base



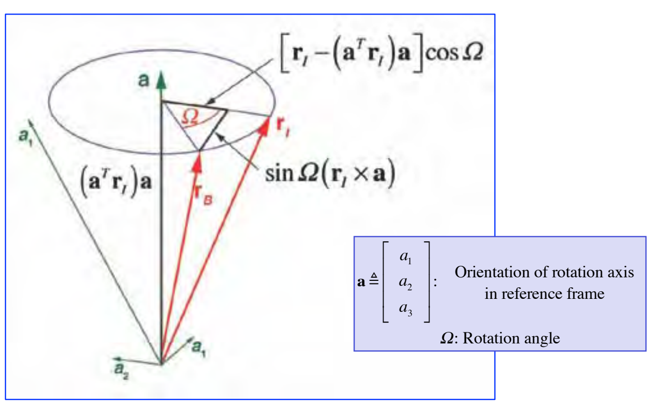
Euler's Rotation Theorem

Vector transformation involves 3 components

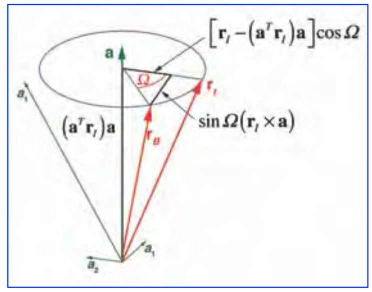
$$\mathbf{r}_B = \mathbf{H}_I^B \mathbf{r}_I$$

$$= \left(\mathbf{a}^T\mathbf{r}_I\right)\mathbf{a} + \left[\mathbf{r}_I - \left(\mathbf{a}^T\mathbf{r}_I\right)\mathbf{a}\right]\cos\Omega + \sin\Omega\left(\mathbf{r}_I \times \mathbf{a}\right)$$

$$= \cos\Omega\, \mathbf{r}_I + \left(1 - \cos\Omega\right)\left(\mathbf{a}^T\mathbf{r}_I\right)\mathbf{a} - \sin\Omega\left(\mathbf{a} \times \mathbf{r}_I\right)$$



Rotation Matrix Derived from Euler's Formula



$$\mathbf{r}_B = \mathbf{H}_I^B \mathbf{r}_I = \cos\Omega\,\mathbf{r}_I + (1-\cos\Omega)\left(\mathbf{a}^T\mathbf{r}_I\right)\mathbf{a} - \sin\Omega\left(\tilde{\mathbf{a}}\mathbf{r}_I\right)$$

Identity

$$\left(\mathbf{a}^T\mathbf{r}_I\right)\mathbf{a} = \left(\mathbf{a}\mathbf{a}^T\right)\mathbf{r}_I$$

Rotation matrix

$$\mathbf{H}_I^B = \cos\Omega\,\mathbf{I}_3 + (1-\cos\Omega)\mathbf{a}\mathbf{a}^T - \sin\Omega\,\tilde{\mathbf{a}}$$

Quaternion Derived from Euler Rotation Angle and Orientation

Quaternion vector

4 parameters based on Euler's formula

$$\mathbf{q} = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix} \triangleq \begin{bmatrix} \mathbf{q}_3 \\ \hdashline q_4 \end{bmatrix} = \begin{bmatrix} \sin(\Omega/2)\mathbf{a} \\ \hdashline \cos(\Omega/2) \end{bmatrix} = \begin{bmatrix} \sin(\Omega/2)\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \\ \hdashline \cos(\Omega/2) \end{bmatrix} \quad (4\times 1)$$

4-parameter representation of 3 parameters; hence, a constraint must be satisfied

$$\mathbf{q}^T\mathbf{q} = q_1^{\,2} + q_2^{\,2} + q_3^{\,2} + q_4^{\,2} = \sin^2(\Omega/2) + \cos^2(\Omega/2) = 1$$

Rotation Matrix Expressed with Quaternion

From Euler's formula

$$\mathbf{H}_I^B = \left[q_4{}^2 - \left(\mathbf{q}_3{}^T \mathbf{q}_3 \right) \right] \mathbf{I}_3 + 2\mathbf{q}_3 \mathbf{q}_3{}^T - 2q_4 \tilde{\mathbf{q}}_3$$

Rotation matrix from quaternion

$$\mathbf{H}_I^B = \begin{bmatrix} q_1^2 - q_2^2 - q_3^2 + q_4^2 & 2\left(q_1 q_2 + q_3 q_4\right) & 2\left(q_1 q_3 - q_2 q_4\right) \\ 2\left(q_1 q_2 - q_3 q_4\right) & -q_1^2 + q_2^2 - q_3^2 + q_4^2 & 2\left(q_2 q_3 + q_1 q_4\right) \\ 2\left(q_1 q_3 + q_2 q_4\right) & 2\left(q_2 q_3 - q_1 q_4\right) & -q_1^2 - q_2^2 + q_3^2 + q_4^2 \end{bmatrix}$$

Quaternion Expressed from Elements of Rotation Matrix

Initialize q(0) from Direction Cosine Matrix or Euler Angles

$$\mathbf{H}_I^B(0) = \begin{bmatrix} h_{11}\left(=\cos\delta_{11}\right) & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix} = \mathbf{H}_I^B\left(\phi_0, \theta_0, \psi_0\right)$$

$$q_4(0) = \frac{1}{2}\sqrt{1 + h_{11}(0) + h_{22}(0) + h_{33}(0)}$$

Assuming that $q_4 \neq 0$

$$\mathbf{q}_3(0) \triangleq \begin{bmatrix} q_1(0) \\ q_2(0) \\ q_3(0) \end{bmatrix} = \frac{1}{4q_4(0)} \begin{bmatrix} \left[h_{23}(0) - h_{32}(0)\right] \\ \left[h_{31}(0) - h_{13}(0)\right] \\ \left[h_{12}(0) - h_{21}(0)\right] \end{bmatrix}$$

Quaternion Vector Kinematics

$$\dot{\mathbf{q}} = \frac{d}{dt}\begin{bmatrix} \mathbf{q}_3 \\ q_4 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} q_4\boldsymbol{\omega}_B - \tilde{\boldsymbol{\omega}}_B\mathbf{q}_3 \\ \text{-----------} \\ -\boldsymbol{\omega}_B{}^T\mathbf{q}_3 \end{bmatrix} \qquad (4\times 1)$$

Differential equation is linear in either q or ω_B

$$\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \\ \dot{q}_4 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 0 & \omega_z & -\omega_y & \omega_x \\ -\omega_z & 0 & \omega_x & \omega_y \\ \omega_y & -\omega_x & 0 & \omega_z \\ -\omega_x & -\omega_y & -\omega_z & 0 \end{bmatrix}_B \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix}$$

Propagate Quaternion Vector Using Body-Axis Angular Rates

$$\begin{bmatrix} p(t) & q(t) & r(t) \end{bmatrix} \triangleq \begin{bmatrix} \omega_x & \omega_y & \omega_z \end{bmatrix}$$

$$\frac{d\mathbf{q}(t)}{dt} = \begin{bmatrix} \dot{q}_1(t) \\ \dot{q}_2(t) \\ \dot{q}_3(t) \\ \dot{q}_4(t) \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 0 & r(t) & -q(t) & p(t) \\ -r(t) & 0 & p(t) & q(t) \\ q(t) & -p(t) & 0 & r(t) \\ -p(t) & -q(t) & -r(t) & 0 \end{bmatrix}_B \begin{bmatrix} q_1(t) \\ q_2(t) \\ q_3(t) \\ q_4(t) \end{bmatrix}$$

Digital integration to compute q(t_k)

$$\mathbf{q}_{\text{int}}(t_k) = \mathbf{q}(t_{k-1}) + \int_{t_{k-1}}^{t_k} \frac{d\mathbf{q}(\tau)}{dt} d\tau$$

Euler Angles Derived from Quaternion

$$\begin{bmatrix} \phi \\ \theta \\ \psi \end{bmatrix} = \begin{bmatrix} \text{atan2}\left\{2\left(q_1 q_4 + q_2 q_3\right), \left[1 - 2\left(q_1^2 + q_2^2\right)\right]\right\} \\ \sin^{-1}\left[2\left(q_2 q_4 - q_1 q_3\right)\right] \\ \text{atan2}\left\{2\left(q_3 q_4 + q_1 q_2\right), \left[1 - 2\left(q_2^2 + q_3^2\right)\right]\right\} \end{bmatrix}$$

- **atan2: generalized arctangent *algorithm,* 2 arguments**
  - **returns angle in proper quadrant**
  - **avoids dividing by zero**
  - **has various definitions, e.g., (MATLAB)**

$$\text{atan2}(y,x) = \begin{cases} \tan^{-1}\left(\dfrac{y}{x}\right) & \text{if } x > 0 \\ \pi + \tan^{-1}\left(\dfrac{y}{x}\right), -\pi + \tan^{-1}\left(\dfrac{y}{x}\right) & \text{if } x < 0 \text{ and } y \geq 0, < 0 \\ \pi/2 \;, -\pi/2 & \text{if } x = 0 \text{ and } y > 0, < 0 \\ 0 & \text{if } x = 0 \text{ and } y = 0 \end{cases}$$